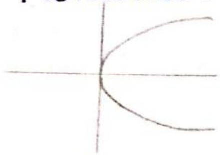
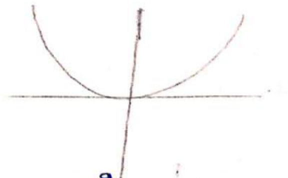
Region for standard curves:

Parabola:

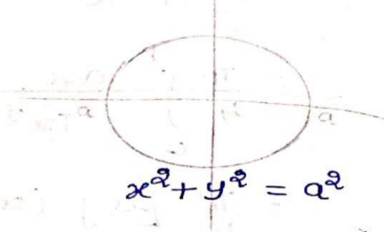


$y^2 = 4ax$



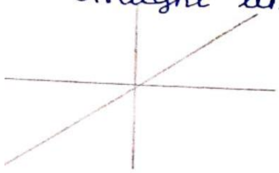
$x^2 = 4ay$

Circle:

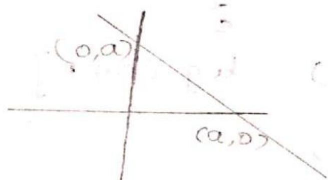


$x^2 + y^2 = a^2$

Straight line:



$x = y$



$x + y = a$

Double Integration based on region:

$\iint dx\,dy \rightarrow$ horizontal strip [left to Right]

$\iint dy\,dx \rightarrow$ vertical strip [down to up]

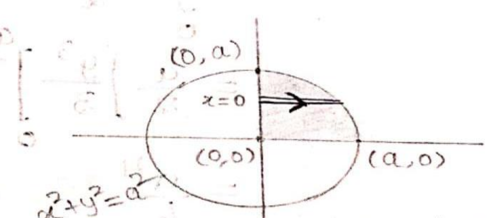
I. Evaluate $\iint_R xy\,dx\,dy$ where R is the 1st quadrant of the circle $x^2 + y^2 = a^2$

Soln. :

Given $x^2 + y^2 = a^2$

$x^2 = a^2 - y^2$

$x = \pm\sqrt{a^2 - y^2}$



At y axis,

Take $x = 0$, $y^2 = a^2$

$y = \pm a$

$dx\,dy$ – horizontal strip

(x limit – interms of y

y limit – constant limits)

Limits :

$x : 0$ to $\sqrt{a^2 - y^2}$ (+ve quadrant)

$y : 0$ to a

Now,

$$\iint_R xy \, dx\, dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy \, dx\, dy$$

$$= \int_0^a \left[\frac{x^2 y}{2} \right]_{x=0}^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{(a^2-y^2)\, y}{2} - 0 \right] dy$$

$$= \int_0^a \frac{(a^2-y^2)\, y}{2} \, dy$$

$$= \frac{1}{2} \int_0^a \left[a^2 y - y^3 \right] dy$$

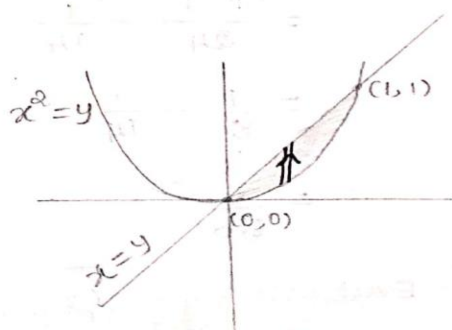
$$= \frac{1}{2} \left[\frac{a^2 y^2}{2} - \frac{y^4}{4} \right]_0^a = \frac{1}{2} \left[\left(\frac{a^4}{2} - \frac{a^4}{4} \right) - 0 \right]$$

$$= \frac{a^4}{2} \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{a^4}{2} \left(\frac{1}{4} \right)$$

$$\iint_R xy \, dx\, dy = \frac{a^4}{8}$$

2]. Evaluate $\iint_R xy(x+y)\, dy\, dx$ over the area between $y = x^2$ and $y = x$.

Soln.:



Given $y = x^2$ and $y = x$.

$\therefore x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, \quad x = 1$

when $x = 0, \quad y = 0$

when $x = 1, \quad y = 1$

Intersection point $(0, 0)$ & $(1, 1)$

$dy\,dx \rightarrow$ vertical strip

y limit $\rightarrow$ in terms of x

x limit $\rightarrow$ constant limits

Limits :

$x : 0$ to 1
$y : x^2$ to x

$$\iint_R xy(x+y)\,dy\,dx = \int_0^1 \int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{y=x^2}^{x} dx$$

$$= \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right)\right] dx$$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \left[\frac{5}{6}\frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_{x=0}^{1}$$

$$\begin{array}{r|l} 2 & 6,\ 14,\ 24 \\ 3 & 3,\ 7,\ 12 \\ & 1,\ 7,\ 4 \end{array}$$

$$= \left(\frac{1}{6} - \frac{1}{14} - \frac{1}{24}\right) - 0$$

$$= \frac{4-1}{24} - \frac{1}{14} = \frac{3}{24} - \frac{1}{14}$$

$$\frac{28 \times 6}{168}$$

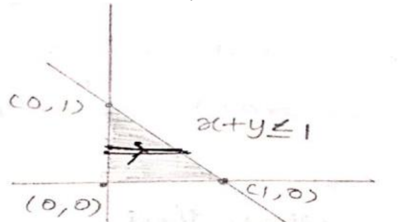
$$= \frac{1}{8} - \frac{1}{14} = \frac{42-24}{336} = \frac{18}{336}$$

$$= \frac{3}{56}$$

3]. Evaluate $\iint_R x^2y\,dx\,dy$ over the region in the +ve quadrant for which $x+y \le 1$

Soln.

Given $x+y=1$



$y=0, \; x=1 \Rightarrow (1,0)$

$x=0, \; y=1 \Rightarrow (0,1)$

Intersection points $(1,0)$ & $(0,1)$

$dx\,dy \to$ horizandal strip

x limit $\to$ interms of y

y limit $\to$ constant limits

Limits :

x : 0 to $1-y$

y : 0 to 1

$$\iint_R x^2 y \, dx\, dy = \int_0^1 \int_0^{1-y} x^2 y \, dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} y\right]_{x=0}^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^3 y}{3} - 0\right] dy$$

$$= \frac{1}{3}\int_0^1 y\left[1-3y+3y^2-y^3\right] dy$$

$$= \frac{1}{3}\int_0^1 \left[y-3y^2+3y^3-y^4\right] dy$$

$$= \frac{1}{3}\left[\frac{y^2}{2}-\frac{3y^3}{3}+\frac{3y^4}{4}-\frac{y^5}{5}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{2}-1+\frac{3}{4}-\frac{1}{5}\right]$$

$$= \frac{1}{3}\left[\frac{20-40+30-8}{40}\right] = \frac{1}{3}\left[\frac{2}{40}\right]$$

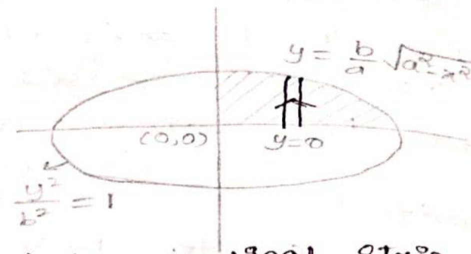
$$\iint_R x^2 y\, dx\, dy = \frac{1}{60}$$

4]. Evaluate $\iint_R x^2 y \, dy \, dx$ lies on the 1st quadrant of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Soln.

Given that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



dydx → vertical strip
y limit → in terms of x
x limit → constant limit

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y^2=\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

At x axis,

$$y=0 \Rightarrow \frac{x^2}{a^2}+0=1$$

$$x^2=a^2$$

$$x=\pm a$$

Limits:

x : 0 to a

y : 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\iint_R x^2 y\,dy\,dx = \int_0^a\int_0^{b/a\sqrt{a^2-x^2}} x^2 y \, dy\, dx$$

$$=\int_0^a x^2\left[\frac{y^2}{2}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$=\int_0^a \frac{x^2}{2}\left[\frac{b^2}{a^2}(a^2-x^2)-0\right]dx$$

$$=\frac{b^2}{2a^2}\int_0^a x^2(a^2-x^2)\,dx$$

$$= \frac{b^2}{2a^2} \int_0^a \left[a^2 x^2 - x^4\right] dx$$

$$= \frac{b^2}{2a^2} \left[a^2 \frac{x^3}{3} - \frac{x^5}{5} \right]_0^a$$

$$= \frac{b^2}{2a^2} \left[\left(\frac{a^5}{3} - \frac{a^5}{5} \right) - 0 \right]$$

$$= \frac{b^2 a^5}{2a^2} \left[\frac{5-3}{15} \right]$$

$$= \frac{a^5 b^2}{2a^2} \left[\frac{2}{15} \right]$$

$$= \frac{a^3 b^2}{15}$$