



UNIT 5 - Multiple Integrals

Triple Integration in Cartesian Coordinates

Triple integration of a function defined over a region is

$$\iiint_R f(x, y, z) dx dy dz$$

Note:

$$\iiint_R dx dy dz \rightarrow \text{Volume of the region } R.$$

Problems:

1. Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2z dz dy dx$.

Solution:

$$\begin{aligned} \int_0^2 \int_1^3 \int_1^2 xy^2z dz dy dx &= \int_0^2 \int_1^3 xy^2 \left(\frac{z^2}{2} \right)_1^2 dy dx \\ &= \int_0^2 \int_1^3 xy^2 \left[\frac{4}{2} - \frac{1}{2} \right] dy dx \\ &= \frac{3}{2} \int_0^2 \int_1^3 xy^2 dy dx = \frac{3}{2} \int_0^2 x \left[\frac{y^3}{3} \right]_1^3 dx \\ &= \frac{1}{2} \int_0^2 x [27 - 1] dx \\ &= \frac{26}{2} \int_0^2 x dx = 13 \left[\frac{x^2}{2} \right]_0^2 \\ &= \frac{13}{2} [4] = 26. \end{aligned}$$

$\int_0^2 \int_1^3 \int_1^2 xy^2z dz dy dx = 26.$



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Q. Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx dy dz$

Solution:

$$\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx dy dz = \int_0^1 \int_0^1 \int_0^1 e^x e^y e^z dx dy dz$$

$$= \int_0^1 e^x dx \int_0^1 e^y dy \int_0^1 e^z dz$$

$$= [e^x]_0^1 [e^y]_0^1 [e^z]_0^1$$

$$= (e^1 - e^0)(e^1 - e^0)(e^1 - e^0)$$

$$= (e - e^0)^3$$

$$= 5.0732$$

Q. Evaluate $\int_0^a \int_0^b \int_0^c xyz dx dy dz$

Solution:

$$\int_0^a \int_0^b \int_0^c xyz dx dy dz = \int_0^a \int_0^b \left(\frac{x^2}{2}\right) yz dy dz$$

$$= \frac{c^2}{2} \int_0^a \int_0^b yz dy dz$$

$$= \frac{c^2}{2} \int_0^a \left(\frac{y^2}{2}\right) z dz$$

$$= \frac{b^2 c^2}{4} \int_0^a z dz$$

$$= \frac{b^2 c^2}{4} \left[\frac{z^2}{2}\right]_0^a = \frac{b^2 c^2}{4} \left[\frac{a^2}{2}\right]$$

$$\int_0^a \int_0^b \int_0^c xyz dx dy dz = \frac{a^2 b^2 c^2}{8} = \frac{1 \cdot 4 \cdot 9}{8} = \frac{9}{2}$$



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4) Evaluate $\int_0^1 \int_0^y \int_0^{x+y} dx dy dz$

Solution:

$$\begin{aligned} \int_0^1 \int_0^y \int_0^{x+y} dx dy dz &= \int_0^1 \int_0^y [z]_0^{x+y} dx dy = \int_0^1 \int_0^y [x+y-0] dx dy \\ &= \int_0^1 \int_0^y (x+y) dx dy = \int_0^1 \left[\frac{x^2}{2} + xy \right]_0^y dy \\ &= \int_0^1 \left[\left(\frac{y^2}{2} + y^2 \right) - 0 \right] dy \\ &= \frac{3}{2} \int_0^1 y^2 dy = \frac{3}{2} \left[\frac{y^3}{3} \right]_0^1 \end{aligned}$$

$$\int_0^1 \int_0^y \int_0^{x+y} dx dy dz = \frac{1}{2}$$

5. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xy dz dx dy$

Solution:

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xy dz dx dy &= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[z \right]_0^{\sqrt{1-x^2-y^2}} dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{1-x^2-y^2}{2} - 0 \right] dy dx \end{aligned}$$



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$$\begin{aligned} &= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} [xy - x^3y - xy^3] dy dx \\ &= \frac{1}{2} \int_0^1 \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \frac{1}{2} \int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3[1-x^2]}{2} - \frac{x[1-x^2]^2}{4} \right] dx \\ &= \frac{1}{2} \int_0^1 \left[\frac{x-x^3}{2} - \frac{x^3-x^5}{2} - \frac{x[1-2x^2+x^4]}{4} \right] dx \\ &= \frac{1}{2} \int_0^1 \left[\frac{x-x^3-x^3+x^5}{2} - x + \frac{2x^2}{4} - \frac{x^5}{4} \right] dx \\ &= \frac{1}{2(4)} \int_0^1 [2(x-2x^3+x^5) - x - x^5 + 2x^3] dx \\ &= \frac{1}{8} \int_0^1 (2x - 4x^3 + 2x^5 - x - x^5 + 2x^3) dx \\ &= \frac{1}{8} \int_0^1 (x - 2x^3 + x^5) dx \\ &= \frac{1}{8} \left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right]_0^1 \\ &= \frac{1}{8} \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right] = \frac{1}{48} \end{aligned}$$
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dx dy dz = \frac{1}{48}$$