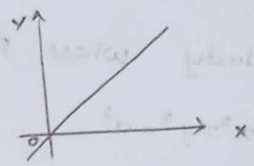
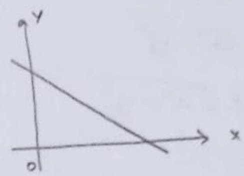
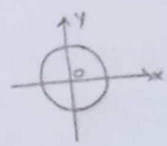
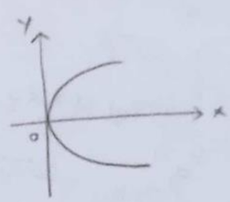
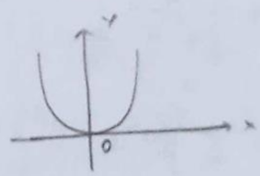
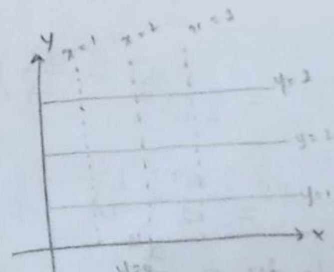
UNIT 5 – Multiple Integrals

Double integration under a given region

Standard diagrams:

Equation	Graph.
1. $y = x$.	
2. $x+y = 1$	
3. $x^2+y^2 = a^2$	
4. $y^2 = 4ax$.	
5. $x^2 = 4ay$	
6. $x=a$, $y=b$	

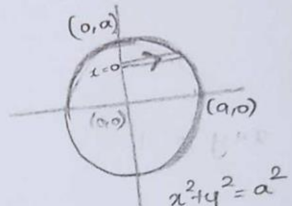
UNIT 5 – Multiple Integrals

Double integration, Based on region

$\iint dx\,dy \rightarrow$ horizontal strip [left to right] x - interms of y, y - constant limit

$\iint dy\,dx \rightarrow$ vertical strip [down to up] $\rightarrow x \rightarrow$ constant limit, $y \Rightarrow$ interms of x

1. Evaluate $\iint_R xy\,dx\,dy$ where R is the 1^{st} quadrant of the circle $x^2+y^2=a^2$



dxdy = horizontal strip

x limit - interms of y

y limit - constant limits

Given : $x^2+y^2=a^2$

$x^2 = a^2-y^2$

$x = \pm\sqrt{a^2-y^2}$

At y axis, take $x=0$, $y^2=a^2$

$y = \pm a$

Limits x : 0 to $\sqrt{a^2-y^2}$

y : 0 to a

Now,

$$\iint_R xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy.$$

$$= \int_0^a \left[\frac{x^2 y}{2}\right]_0^{\sqrt{a^2-y^2}} = \int_0^a \left[\frac{(a^2-y^2)y}{2} - 0\right] dy$$

$$= \frac{1}{2}\int_0^a (a^2 y - y^3)\,dy = \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - 0\right] = \frac{a^4}{2}\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$= \frac{a^4}{2}\left(\frac{1}{4}\right)$$

$$\iint_R xy\,dx\,dy = \frac{a^4}{8}$$

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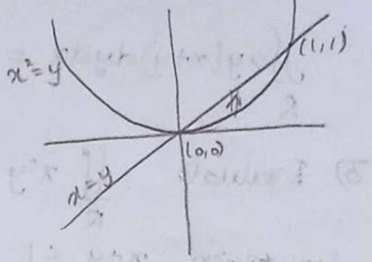
2. Evaluate $\iint_R xy(x+y)\,dy\,dx$ over the area between $y = x^2$ and $y = x$



dydx → vertical strip
y limit → in terms of x
x limit → constant limit

Given: $y = x^2$ and $y = x$.

$\therefore\ x^2 = x$

$x^2 - x = 0 \Rightarrow x(x-1) = 0$

$x = 0$ and $x = 1$

when $x = 0,\ y = 0$

when $x = 1,\ y = 1$

Intersection point $(0,0)$ & $(1,1)$

Limits: x : 0 to 1

y : x^2 to x

$$\iint_R xy(x+y)\,dy\,dx = \int_0^1\int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1\int_{x^2}^{x} (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{y=x^2}^{x} dx$$

$$= \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right)\right] dx$$

$$= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx$$

$$= \left[\frac{5}{6}\cdot\frac{x^5}{5} - \frac{1}{2}\frac{x^7}{7} - \frac{1}{3}\frac{x^8}{8}\right]_0^1$$

$$= \left[\frac{1}{6}(1) - \frac{1}{14}(1) - \frac{1}{24}(1)\right] - 0$$

$$= \frac{4-1}{24} - \frac{1}{14} = \frac{3}{24} - \frac{1}{14} = \frac{1}{8} - \frac{1}{14} = \frac{42-24}{336}$$

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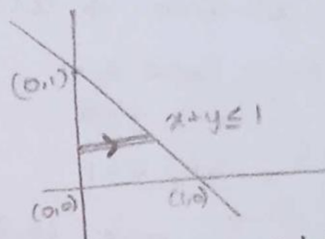
$$= \frac{18}{336}$$

$$\iint_R xy(x+y)\,dy\,dx = \frac{3}{56}$$

3) Evaluate $\iint_R x^2y\,dx\,dy$ over the region in the +ve quadrant for which $x+y \le 1$



Given $x+y=1$

$y=0\ ,\ x=1 \Rightarrow (1,0)$

$x=0\ ,\ y=1 \Rightarrow (0,1)$

Intersection points $(1,0)$ & $(0,1)$

$dx\,dy \to$ horizontal strip
x limit $\to$ in terms of y
y limit $\to$ constant limits

Limits:- x : 0 to $1-y$
y : 0 to 1

$$\iint_R x^2y\,dx\,dy = \int_0^1\int_0^{1-y} x^2y\,dx\,dy = \int_0^1 \left[\frac{x^3}{3}y\right]_{x=0}^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^3}{3}y - 0\right] dy = \frac{1}{3}\int_0^1 y(1-3y+3y^2-y^3)\,dy$$

$$= \frac{1}{3}\int_0^1 \left[y-3y^2+3y^3-y^4\right] dy$$

$$= \frac{1}{3}\left[\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right] = \frac{1}{3}\left[\frac{20-40+30-8}{40}\right]$$

$$= \frac{1}{3}\left[\frac{2}{40}\right]$$

$$\iint_R x^2y\,dx\,dy = \frac{1}{60}.$$

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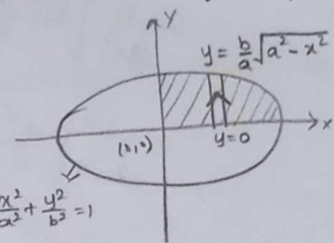
4. Evaluate $\iint_R x^2y \, dy\,dx$ lies on the 1^{st} quadrant of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$dy\,dx \to$ Vertical strip

y limit $\to$ interms of x

x limit $\to$ constant limit

Given that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

At x axis

$$y=0 \Rightarrow \frac{x^2}{a^2}+0=1$$

$$x^2=a^2$$

$$x=\pm a$$

Limits : $x=0$ to a

$y : 0$ to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\iint_R x^2y\,dy\,dx = \int_0^a\int_0^{b/a\sqrt{a^2-x^2}} x^2y\,dy\,dx$$

$$= \int_0^a x^2\left[\frac{y^2}{2}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x^2}{2}\left[\frac{b^2}{a^2}(a^2-x^2)-0\right]dx$$

$$= \frac{b^2}{2a^2}\int_0^a x^2(a^2-x^2)\,dx$$

$$= \frac{b^2}{2a^2}\int^a (a^2x^2-x^4)\,dx$$

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$$= \frac{b^2}{2a^2}\left[a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left[\left(\frac{a^5}{3} - \frac{a^5}{5}\right) - 0\right]$$

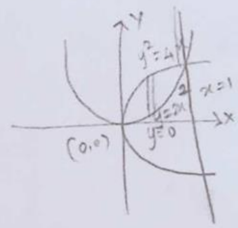
$$= \frac{b^2a^5}{2a^2}\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{b^2a^3}{2}\left[\frac{5-3}{15}\right] = \frac{b^2a^3}{2}\left[\frac{2}{15}\right]$$

$$= \frac{a^3b^2}{15}$$

$$\iint_R x^2y \, dy\,dx = \frac{a^3b^2}{15}$$

(4) Using double integration find the area enclosed by the curves $y = 2x^2$ & $y^2 = 4x$.



$y = 2x^2 \rightarrow (1)$

$y^2 = 4x \rightarrow (2)$

Sub (1) in (2)

$(2x^2)^2 = 4x$

$4x^4 = 4x$

$x^4 - x = 0 \quad x(x^3-1) = 0$

$x = 0, \quad x^3 = 1 \Rightarrow x = 1$

when $x = 0, y = 0$

$x = 1, y = 2$

$x : 0$ to 1

$y : 2x^2$ to $2\sqrt{x}$

$$A = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy\,dx$$

$$= \int_0^1 [2\sqrt{x} - 2x^2]\,dx$$

$$= 2\left[\frac{x^{3/2}}{3/2} - \frac{2x^3}{3}\right]_0^1$$

$$= 2/3$$