UNIT 5 – Multiple Integrals

Applications of multiple integrals are to find areas and volume of various bodies just by taking a little part of them into consideration.

In probability theory, it is used to evaluate probabilities of two dimensional continuous random variables

Double Integration (cartesian coordinate)

A double integral is computed by repeated single variable integration, integrate wrt one variable treating the other variable as constant.

$$I = \iint_R f(x,y)\,dx\,dy$$ where R is the Region.

Formulae:

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. $\int k\,dx = kx$
3. $\int \frac{1}{x} dx = \log x$
4. $\int e^x dx = e^x$
5. $\int \sin x\, dx = -\cos x$
6. $\int \cos x\, dx = \sin x$
7. $\int \tan x\, dx = \log(\sec x)$ (or) $-\log(\cos x)$
8. $\int \cot x\, dx = \log(\sin x) = \log(\csc x)$
9. $\int \operatorname{cosec} x\, dx = \log[\operatorname{cosec} x - \cot x]$
10. $\int \sec x\, dx = \log[\sec x + \tan x]$
11. $\int \sec^2 x\, dx = \tan x$
12. $\int \operatorname{cosec}^2 x\, dx = -\cot x$.

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13) $\int \sec x \tan x \, dx = \sec x$

14. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$

15. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left[\frac{x}{a}\right]$

16. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

17. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$

19. $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x$

20. $\int \frac{1}{1+x^2} dx = \tan^{-1} x$

21. $\int \frac{-1}{1+x^2} dx = \cot^{-1} x$

22. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$

23. $\int \frac{-1}{x\sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x$

24. $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

25. $\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left[x+\sqrt{x^2+a^2}\right]$

26. $\int u\,dv = uv - \int v\,du$

27. Bernoulli's formula. $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

## Problems on double integration in cartesian coordinates

II 1. Evaluate $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$

Soln:-

Now $\int_0^1 \int_1^2 x(x+y)\,dy\,dx = \int_0^1 \left[\int (x^2+xy)\,dy\right] dx$

$$= \int_0^1 \left[x^2 y + x\frac{y^2}{2}\right]_{y=1}^{2} dx$$

$$= \int_0^1 \left[\left(2x^2+\frac{4x}{2}\right) - \left(x^2+\frac{x}{2}\right)\right] dx$$

$$= \int_0^1 \left[2x^2+2x-x^2-\frac{x}{2}\right] dx$$

$$= \int_0^1 \left[x^2+\frac{3}{2}x\right] dx$$

$$= \left[\frac{x^3}{3}+\frac{3}{2}\,\frac{x^2}{2}\right]_{x=0}^{1} = \left[\left(\frac{1}{3}+\frac{3}{4}\right)-0\right]$$

$$\therefore \int_0^1\int_1^2 x(x+y)\,dy\,dx = \frac{13}{12}$$

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2. Evaluate $\int_0^a \int_0^b xy(x-y)\,dx\,dy$

Soln:- $\int_0^a \int_0^b xy(x-y)\,dx\,dy = \int_0^a \int_0^b [x^2y - xy^2]\,dx\,dy$

$$= \int_0^a \left[\frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_{x=0}^{b} dy$$

$$= \int_0^a \left[\frac{b^3y}{3} - \frac{b^2y^2}{2}\right] dy$$

$$= \left[\frac{b^3}{3}\left(\frac{y^2}{2}\right) - \frac{b^2}{2}\left(\frac{y^3}{3}\right)\right]_{y=0}^{a}$$

$$= \left[\frac{b^3a^2}{6} - \frac{b^2a^3}{6}\right] - 0$$

$$\int_0^a \int_0^b xy(x-y)\,dx\,dy = \frac{a^2b^2}{6}[b-a]$$

3. Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy$

Soln:-
Now $\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy = \int_2^3 \int_1^2 \frac{1}{x}\cdot\frac{1}{y}\,dx\,dy$

$$= \int_2^3 \frac{1}{y}[\log x]_{x=1}^{2}\,dy$$

$$= \int_2^3 \frac{1}{y}[\log 2 - \log 1]\,dy$$

$$= \int_2^3 \frac{1}{y}\log 2\,dy \qquad [\because \log 1 = 0$$

$$= \log 2 \int_2^3 \frac{dy}{y} = \log 2\,[\log y]_2^3$$

$$= \log 2[\log 3 - \log 2] = \log 2 \log\left(\frac{3}{2}\right)$$

$$\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy = \log 2 \log\left(\frac{3}{2}\right)$$

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4. Evaluate $\int_0^3\int_0^2 e^{x+y}\,dy\,dx$.

$$\int_0^3\int_0^2 e^{x+y}\,dy\,dx = \int_0^3\int_0^2 e^x\cdot e^y\,dy\,dx = \int_0^3 e^x\left[e^y\right]_{y=0}^{2}dx$$

$$= \int_0^3 e^x\left[e^2-e^0\right]dx = \left[e^2-1\right]\left[e^x\right]_{x=0}^{3}$$

$$= \left[e^2-1\right]\left[e^3-e^0\right] = \left[e^2-1\right]\left[e^3-1\right]$$

$$\therefore \int_0^3\int_0^2 e^{x+y}\,dy\,dx = \left[e^2-1\right]\left[e^3-1\right]$$

5. Evaluate $\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Soln:- $$\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy = \int_0^5\int_0^{x^2}\left[x^3+xy^2\right]dy\,dx$$

$$= \int_0^5\left[x^3y + x\frac{y^3}{3}\right]_{y=0}^{x^2}dx$$

$$= \int_0^5\left[\left(x^5+\frac{x^7}{3}\right)-(0+0)\right]dx$$

$$= \int_0^5\left[x^5+\frac{x^7}{3}\right]dx$$

$$= \left[\frac{x^6}{6}+\frac{x^8}{24}\right]_0^5$$

$$= \left(\frac{5^6}{6}+\frac{5^8}{24}\right)-0$$

$$= 5^6\left[\frac{29}{24}\right]$$

$$\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy = 5^6\left[\frac{29}{24}\right]$$

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6. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$

Soln:-
$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \int_0^a [y]_{y=0}^{\sqrt{a^2-x^2}} dx$$
$$= \int_0^a \sqrt{a^2-x^2}\,dx$$
$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$
$$= \left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) - (0+0)$$
$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}.$$

7. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

Soln:-
$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\,dx$$
$$= \int_0^1 \int_0^1 \frac{dy}{y^2+\left(\sqrt{1+x^2}\right)^2}\,dx$$
$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left[\frac{y}{\sqrt{1+x^2}}\right]\right)_{y=0}^{\sqrt{1+x^2}} dx$$
$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - 0\right] dx$$
$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\frac{\pi}{4}\right) dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}$$
$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1^2}\right)\right]_0^1$$

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$$= \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log(0+1)\right]$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

8. Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy \, dx\, dy$

Soln:-

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx\, dy = \int_0^a \left[\frac{x^2}{2} y\right]_{x=0}^{\sqrt{ay}} dy$$

$$= \int_0^a \left[\frac{ay}{2} y - 0\right] dy$$

$$= \int_0^a \frac{ay^2}{2} \, dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{a}{2}\left[\frac{a^3}{3}\right]$$

$$= \frac{a^4}{6}$$

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx\, dy = \frac{a^4}{6}.$$