UNIT 5 – Multiple Integrals

Change of order of Integration.

1. Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dy\,dx$ using change of order of Integration.

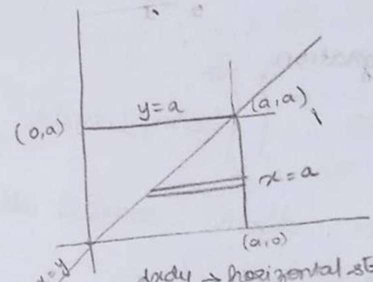
Given:- $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dy\,dx = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\,dy$ [correct form]

order : $dx\ dy$

Limits :

$x :\ y$ to a

$y :\ 0$ to a



dxdy → horizontal strip
x limit → interms of y
y limit → constant limits

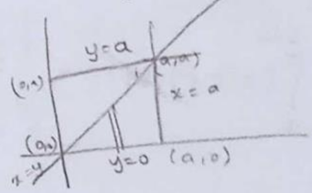
After changing the order of Integration,

Now order : $dy\,dx$ [vertical strip]

Now limits :

$x :\ 0$ to a

$y :\ 0$ to x



$$\therefore \int_0^a \int_y^a \left[\frac{x}{x^2+y^2}\right] dy\,dx = \int_0^a \int_0^x \frac{dy}{y^2+x^2}\, x\,dx$$

$$= \int_0^a \left[\int_0^x \frac{dy}{y^2+x^2}\right] x\,dx$$

$$= \int_0^a \left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_{y=0}^{x} x\,dx$$

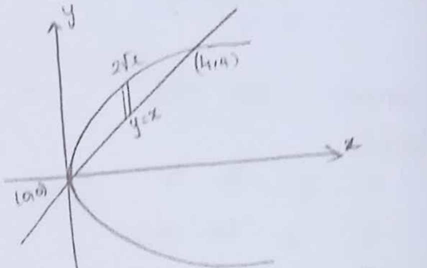
$$= \int_0^a \left[\frac{1}{x}\tan^{-1}(1) - 0\right] x\,dx$$

UNIT 5 – Multiple Integrals

$$= \frac{\pi}{4}\int_0^a dx = \frac{\pi}{4}\left[x\right]_0^a$$

$$= \frac{\pi a}{4}$$

2. Evaluate $\int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\,dy\,dx$ using change of order of integration.



Given: $\int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\,dy\,dx$

order: $dy\,dx$ (vertical strip)

Given $y=x$ & $y=2\sqrt{x}$

$x = 2\sqrt{x}$ $\quad y^2=4x$

$x^2 = 4x \Rightarrow x^2-4x=0$

$x(x-4)=0$

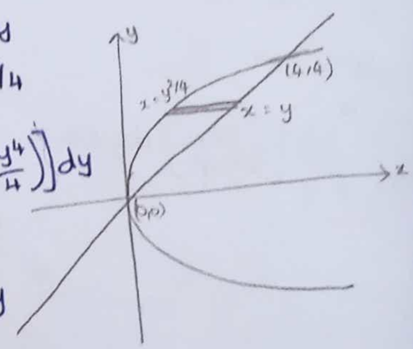
$\Rightarrow x=0,\ x=4$

Intersection pts $(0,0)$ $(4,4)$

x : $y^2/4$ to y

y : 0 to 4

$$\therefore \int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\,dy\,dx = \int_0^4 \int_{y^2/4}^{y} (x^2+y^2)\,dx\,dy$$



$$= \int_0^4 \left[\frac{x^3}{3} + y^2x\right]_{x=y^2/4}^{y} dy$$

$$= \int_0^4 \left[\left(\frac{y^3}{3}+y^3\right) - \left(\frac{1}{3}\frac{y^6}{64}+\frac{y^4}{4}\right)\right] dy$$

$$= \int_0^4 \left[\frac{4}{3}y^3 - \frac{1}{192}y^6 - \frac{y^4}{4}\right] dy$$

$$= \left[\frac{4}{3}\cdot\frac{y^4}{4} - \frac{1}{192}\frac{y^7}{7} - \frac{y^5}{5\times 4}\right]_0^4 = \left[\frac{1}{3}\cdot 4^4 - \frac{1}{1344}4^7 - \frac{1}{20}4^5\right] - 0$$

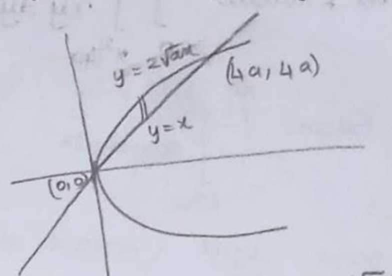
$$= 4^4\left[\frac{1}{3} - \frac{4^3}{1344} - \frac{4}{20}\right] = 256\left[\frac{1}{3} - \frac{64}{1344} - \frac{1}{5}\right]$$

$$= 256\left[\frac{1}{3} - \frac{1}{21} - \frac{1}{5}\right] = 256\left[\frac{35-5-21}{105}\right] = \frac{2304}{105} = \boxed{\frac{768}{35}}$$

UNIT 5 – Multiple Integrals

3. Evaluate $\int_0^{4a}\int_x^{2\sqrt{ax}} xy\, dy\, dx$ using change of order of Integration

Given:- $\int_0^{4a}\int_x^{2\sqrt{ax}} xy\, dy\, dx$



order: $dy\,dx$ (vertical strip)

Limits: x : 0 to $4a$

y : x to $2\sqrt{ax}$

$y = x$ & $y = 2\sqrt{ax} \Rightarrow x = 2\sqrt{ax}$

$y^2 = 4ax$ $\quad x^2 = 4ax$

$x^2 - 4ax = 0$

$x(x-4a) = 0$

$x = 0, x = 4a$

When $x = 0, y = 0$

$x = 4a, y = 4a$

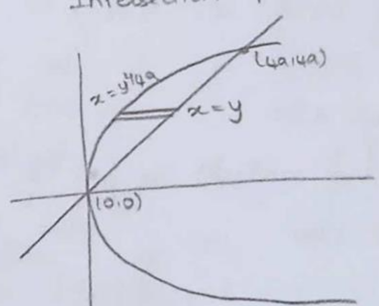
Intersection pts $(0,0)$ & $(4a, 4a)$

After changing the order of integration,

Now order : $dx\,dy$ (Horizontal strip)

x : $y^2/4a$ to y

y : 0 to $4a$



$$\therefore \int_0^{4a}\int_x^{2\sqrt{ax}} xy\, dy\, dx = \int_0^{4a}\int_{y^2/4a}^{y} xy\, dx\, dy$$

$$= \int_0^{4a}\left[\int_{y^2/4a}^{y} x\, dx\right] y\, dy$$

$$= \int_0^{4a}\left[\frac{x^2}{2}\right]_{y^2/4a}^{y} y\, dy$$

$$= \frac{1}{2}\int_0^{4a}\left[y^2 - \frac{y^4}{16a^2}\right] y\, dy = \frac{1}{2}\int_0^{4a}\left[y^3 - \frac{y^5}{16a^2}\right] dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{6\times 16a^2}\right]_0^{4a} = \frac{1}{2}\left[\frac{(4a)^4}{4} - \frac{(4a)^6}{6\times 16a^2}\right]$$

$$= \frac{1}{2}\left[\frac{256a^4}{4} - \frac{4096a^6}{96a^2}\right] = \frac{a^4}{2}\left[64 - \frac{128}{3}\right]$$

$$= \frac{a^4}{2}\left[\frac{192-128}{3}\right] = a^4\left[\frac{64}{6}\right]$$

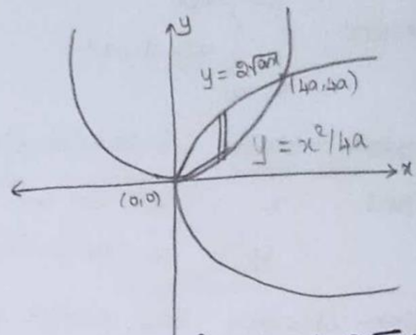
$$\int_0^{4a}\int_x^{2\sqrt{ax}} xy\, dy\, dx = \frac{32a^4}{3}$$

UNIT 5 – Multiple Integrals

4) Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy dx$ using change of order of integration

Given: $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy dx$

$a = 1$



Order: $dydx$ (vertical strip)

Limits: x : 0 to $4a$

y : $\frac{x^2}{4a}$ to $2\sqrt{ax}$

Given $y = x^2/4a$ & $y = 2\sqrt{ax}$

$x^2 = 4ay \qquad y^2 = 4ax$

Now

$\frac{x^2}{4a} = 2\sqrt{ax}$

$\frac{x^4}{16a^2} = 4ax$

$x^4 = 64a^3x$

$x^4 - 64a^3x = 0$

$x(x^3 - 64a^3) = 0$

$x = 0, \; x^3 = 64a^3$

$x = 4a$

when $x = 0$, $y = 0$

$x = 4a$, $y = \frac{16a^2}{4a}$

$= 4a$

Intersection pts are $(0,0)$ & $(4a, 4a)$

After changing the order of integration

Now order: $dxdy$ (horizontal strip)

Now limits x: $y^2/4a$ to $2\sqrt{ay}$

y: 0 to $4a$

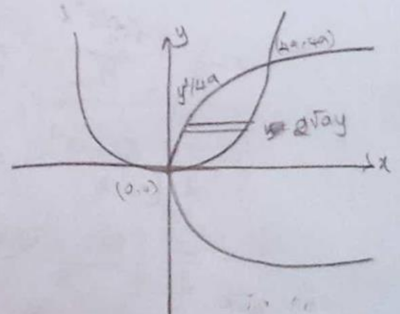
$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\, dx dy$$

$$= \int_0^{4a}\left[\frac{x^2}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} y\,dy$$

$$= \frac{1}{2}\int_0^{4a}\left[4ay - \frac{y^4}{16a^2}\right] y\,dy.$$

$$= \frac{1}{2}\int_0^{4a}\left[4ay^2 - \frac{y^5}{16a^2}\right] dy$$

$$= \frac{1}{2}\left[4a\frac{y^3}{3} - \frac{y^6}{16a^2 \times 6}\right]_0^{4a}$$



UNIT 5 – Multiple Integrals

$$= \frac{1}{2}\left[\left(4a\,\frac{(4a)^3}{3} - \frac{(4a)^6}{96a^2}\right) - 0\right]$$

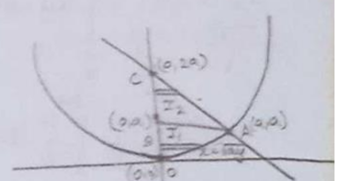
$$= \frac{1}{2}\left[\frac{4^4a^4}{3} - \frac{4^6a^6}{96a^2}\right] = \frac{4^4}{2}\left[\frac{a^4}{3} - \frac{4^2a^4}{96}\right]$$

$$= \frac{4^4a^4}{2}\left[\frac{1}{3} - \frac{16}{96}\right] = \frac{256a^4}{2}\left[\frac{1}{3} - \frac{1}{6}\right]$$

$$= \frac{256\,a^4}{2}\left[\frac{1}{6}\right] = \frac{64}{3}a^4$$

5. Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ using change of order of Integration

Solution:- Given $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$



order : $dy\,dx$ [vertical strip)

Limits x : 0 to a

y : x^2/a to $2a-x$

Given $y = \frac{x^2}{a}$ & $y = 2a - x$

$x^2 = ay$ $\quad x + y = 2a$

when $x = 0$, $y = 2a$

$x = a$ $\quad y = a$

Intersection pts

$(0,2a)$ & (a,a)

After changing the order of integration

Now order : $dx\,dy$

The region of integration can be split up into two portions

i.e) i) OAB ii) ABC and

$I = I_1 + I_2$

i) In the region OAB

limits x : 0 to $\sqrt{ay}$

y : 0 to a

ii) In the region ABC

limits x : 0 to $2a-y$

y : a to $2a$.

UNIT 5 – Multiple Integrals

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy = \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y\, dy$$

$$= \int_0^a \left[\frac{ay}{2} - 0\right] y\, dy = \int_0^a \frac{ay^2}{2} dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a \Rightarrow \frac{a^4}{6}$$

$$I_1 = \frac{a^4}{6} \rightarrow ①$$

$$I_2 = \int_a^{2a} \int_0^{2a-y} xy\, dx\, dy = \int_0^{2a} \left[\frac{x^2}{2}\right]_0^{2a-y} y\, dy$$

$$= \frac{1}{2}\int_0^{2a} [2a-y]^2 y\, dy = \frac{1}{2}\int_0^{2a} [4a^2+y^2-4ay] y\, dy.$$

$$= \frac{1}{2}\int_0^{2a} [4a^2 y + y^3 - 4ay^2]\, dy$$

$$= \frac{1}{2}\left[4a^2\frac{y^2}{2} + \frac{y^4}{4} - 4a\frac{y^3}{3}\right]_a^{2a}$$

$$= \frac{1}{2}\left[\left[2a^2[2a]^2 + \frac{(2a)^4}{4} - \frac{4a}{3}(2a)^3\right] - \left[2a^2(a^2) + \frac{a^4}{4} - \frac{4a^4}{3}\right]\right]$$

$$= \frac{1}{2}\left[8a^4 + \frac{16a^4}{4} - \frac{32}{3}a^4 - 2a^4 - \frac{a^4}{4} + \frac{4}{3}a^4\right]$$

Chk

$$= \frac{a^4}{2}\left[10 - \frac{1}{4} - \frac{28}{3}\right] = \frac{a^4}{2}\left[\frac{+5}{12}\right]$$

$$I_2 = \frac{5a^4}{24} \rightarrow ②$$

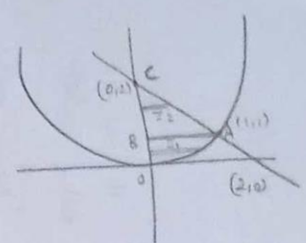
From ① & ② $\quad I = I_1 + I_2 = \frac{a^4}{6} + \frac{5a^4}{24} = \frac{9a^4}{24}$

$$\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx = \frac{3}{8}a^4.$$

UNIT 5 – Multiple Integrals

6. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ using change of order of Integration

Solution:- $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$



order : $dydx$

Limits x : 0 to 1

y : x^2 to $2-x$

Given $y=x^2$ & $y=2-x$

$x^2=y$ $\quad x+y=2$

$x^2=2-x$

$x^2+x-2=0$

$\Rightarrow (x+2)(x-1)=0$

$x=-2,1$

when $x=1$, $y=1$

After changing the order of integration

Now order : $dxdy$ (horizontal strip)

The region of integration can be split up into two portions

i) OAB ii) ABC ie) $I=I_1+I_2$

i) In the region OAB

Limits : x : 0 to $\sqrt{y}$

y : 0 to 1

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\,dxdy = \int_0^1 \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y\,dy$$

$$= \frac{1}{2}\int_0^1 [y]\,y\,dy = \frac{1}{2}\int_0^1 y^2 dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$I_1 = \frac{1}{6} \longrightarrow (1)$$

(ii) In the region ABC,

limits : x : 0 to $2-y$

y : 1 to 2

$$I_2 = \int_1^2 \int_0^{2-y} xy\,dxdy = \int_1^2 \left[\frac{x^2}{2}\right]_0^{2-y} y\,dy = \frac{1}{2}\int_1^2 [2-y]^2 y\,dy$$

UNIT 5 – Multiple Integrals

$$= \frac{1}{2}\int_1^2 (4-4y+y^2)y\,dy = \frac{1}{2}\int_1^2 (4y-4y^2+y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[2(4)+\frac{16}{4}-\frac{4}{3}(8)-\left(2(1)+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[8+4-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right]$$

$$= \frac{1}{2}\left[10-\frac{28}{3}-\frac{1}{4}\right] = \frac{1}{2}\left[\frac{5}{12}\right]$$

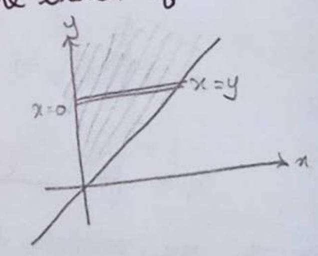
$$I_2 = \frac{5}{24} \rightarrow (2)$$

From (1) & (2) $\quad I = I_1 + I_2 = \frac{1}{6}+\frac{5}{24} = \frac{4+5}{24} = \frac{9}{24}$

$$I = \frac{3}{8}$$

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \frac{3}{8}$$

7. Evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration



Given: $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

order: $dy\,dx$ (vertical strip)

Limits: $x : 0$ to ∞

$y : x$ to ∞

After changing the order of integration

Now order : $dx\,dy$ (horizontal strip)

Now limits $\quad x : 0$ to y

$y : 0$ to ∞

UNIT 5 – Multiple Integrals

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y \, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -\left[e^{-\infty} - 1\right] = -[0-1]$$

$$= 1$$

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = 1$$