MAXWELL'S EQUATION

Maxwell's equation forms the foundation of electromagnetic theory.

There are four Maxwell's Equation in EMT.

First two equations are known as steady state equations.

Last two equations are known as time varying equations.

Maxwell's Equation-I

(From Gauss's Law)

Integral Form:

States that the total electric flux through any closed surface is equal to the charge enclosed by it.

According to Gauss Law

$\oiint_s \vec{E} ds = \frac{q}{\epsilon}$------------(1)

$\oiint_s \vec{E} \epsilon ds = q$

$\oiint_s \vec{D} ds = q$----------(2) $\quad$ [$D = \epsilon_0 \vec{E}$]

If ρ be the charge density,then total charge inside the closed surface is given by

$q = \iiint_v \rho dv$ --------(3)

substituting eqn(3) in eqn(2)

$$\boxed{\oiint_s \vec{D} ds = \iiint_v \rho dv \text{ -------(4)}}$$

This is Maxwell's Equation in Integral Form.

Gauss's Law in electrostatics:

Applying Gauss's divergence theorem to L.H.S of eqn(4)

i.e,

$\oiint_s \vec{D}\mathrm{ds} = \iiint \vec{\Delta}.\vec{D}\mathrm{dv}$ ------ (5)

substituting eqn(5) in eqn(4)

$\iiint_v \vec{\Delta}.\vec{D}\mathrm{dv} = \iiint \rho dv$---------(6)

$$\boxed{\vec{\Delta}.\vec{D}=\rho}$$

$\vec{\Delta}.\in_0 \ \vec{\ } = \rho$

$\in_0 \vec{\Delta}.\vec{E} = \rho$

$\vec{\Delta}.\ \vec{E} = \frac{\rho}{\in_0}$

This is Maxwell's Equation from Gauss's law in electrostatics in differential form.

Statement:

The total electric displacement through the surface enclosing a volume is equal to the total charge with in the volume.

<u>Maxwell's Equation-II</u>

(From Gauss's law in magnetostatics)

<u>Integral Form:</u>

It is well known fact in magnetism that the magnetic lines of force are continous and do not appear to have the origin or the end.Thus the total magnetic flux through any closed surface in a magnetic field is zero.

$$\boxed{\oiint \vec{B}\ .\ \vec{ds} = 0\text{-----------(1)}}$$

This is Maxwell's Equation in integral form,from Gauss's law in magnetostatics.

Applying Gauss divergence theorem to the L.H.S of eqn(1) ,we get

$\oiint \vec{B} \cdot \vec{ds} = \iiint_v \vec{\Delta} . \vec{B} dv$------(2)

on substituting eqn (2) in eqn(1),we have

$\iiint_v \vec{\Delta} . \vec{B} dv$ = 0 ---------(3)

(or)

$\boxed{\vec{\Delta} . \vec{B} = 0}$

This is Maxwell's Equation in differential form,from Gauss's law in magnetostatics.

Statement:

The net magnetic flux emerging through any losed surface is zero.

<u>Maxwell's Equation-III</u>

(From Faraday's Law)

Magnetic flux through a small area

ds = $\vec{B} . \vec{ds}$ ------- (1)

Therefore magnetic flux

$\varphi_B = \oiint_S \vec{B} . \vec{ds}$------(2)

Faraday's Law states that the induced emf e is the rate of change of magnetic flux φ_B

Therefore 'e '= (-) $\frac{d\varphi_B}{dt}$

=(-) $\frac{d}{dt}[\oiint_S \vec{B} . \vec{ds}]$

= $\oiint_S \frac{d\vec{B}}{d} s$ ------------(3)

If $\vec{E}$ be the electric field strength,then we know that $\vec{E} = \frac{dv}{dl}$

dv = $\vec{E}$.dl

v = $\int dv$ = $\int \vec{E}.\vec{dl}$

v = e = $\int \vec{E}.\vec{dl}$

e = $\oint \vec{E}.\vec{dl}$ -------(4)

Equating (3) and (4)

$$\boxed{\oint \vec{E}.\vec{dl} = -\oiint_S \frac{dB}{dt} ds \text{--------------(5)}}$$

The is Maxwell's Equation in integral form,from Faraday's law of electromagnetic induction.

Applying Stoke's theorem to L.H.S of eqn(5)

$\oint_C \vec{E}.\vec{dl} = \iint (\vec{\Delta} X \vec{E})$ ds-------- (6)

on substituting (6) in (5) we get

$\iint_S (\vec{\Delta} X \vec{E})$ ds = -$\iint_S \frac{d\vec{B}}{dt}.ds$ -------(7)

it must be true for all surface S,

$$\boxed{(\vec{\Delta} X \vec{E}) = (-) \frac{d\vec{B}}{dt} \text{----------------(8)}}$$

(8) represents Maxwell's Equation from Faraday's law.

<u>Maxwell's Equation-IV</u>

(From Ampere's Circuital Law)

Ampere's Law states that the line integral of magnetic field intensity H on any closed path is equal to the current (I) enclosed by the path.

$\oint \vec{H}.\vec{dl}$ = I ---------- (1)

But current density J = $\frac{I}{A}$

where A- Cross Sectional Area

I = JA

since A = $\iint_S \ ds$

I = J $\iint_S \ ds$

I = $\iint_S$ Jds----------(2)

substituting (2) in (1) we have

$\oint \vec{H}\,\vec{dl}$ = $\iint_S$ Jds --------- (3)

Ampere's Law is modified by displacement current density

$\oint \vec{H}\,.\vec{dl}$ =($\vec{J_C} + \vec{J_D}$) ds----------(4)

Substituting $\vec{J}_C$ = $\sigma\vec{E}$ and $\vec{J}_D$ = $\frac{d\vec{D}}{dt}$ we have

$$\oint \vec{H}\,.\vec{dl} = \iint_S \left(\sigma\vec{E} + \frac{d\vec{D}}{dt}\right)ds$$

$$\oint \vec{H}\,.\vec{dl} = \iint_S \left(\sigma\vec{E} + \frac{\in d\vec{E}}{dt}\right)ds$$

J stands for conduction current density

$$\oint \vec{H}\,.\vec{dl} = \iint_S \left(\vec{J} + \frac{d\vec{D}}{dt}\right)ds\text{----------(6)}$$

This is Maxwell's Equation in integral form from Ampere's circuital Law.

Applying Stoke's theorem to L.H.S of eqn (6)we have

$\oint \vec{H}\,.\vec{dl} = \iint_S (\vec{\nabla}\ X\ \vec{H})s$ ----------(7)

sub (7) in (6)

$$\iint_S (\vec{\nabla}\ X\ \vec{H})s = \iint_S \left(\vec{J} + \frac{d\vec{D}}{dt}\right)ds\text{--------(8)}$$

$$(\vec{\nabla}\ \mathrm{X}\ \vec{H}) = \left(\vec{J} + \frac{d\vec{D}}{dt}\right)\text{ -------- (9)}$$

$$(\vec{\nabla}\ \mathrm{X}\ \vec{H}) = \left(\sigma\vec{E} + \frac{\in d\vec{E}}{dt}\right)\text{ ------- (10)}$$

eqn (9) and (10) are Maxwell's Equation in differential form from Ampere's circuital Law.

Wave Equation for electric field vector($\vec{E}$)

$(\vec{\nabla} \times \vec{E}) = (-) \frac{d\vec{B}}{dt}$ ------------------(A)

Taking the curl on both sides of equation

$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times (- \frac{d\vec{B}}{dt})$

$= (-) \frac{d}{dt} (\vec{\nabla} \times \vec{B})$

$= (-) \frac{d}{dt} (\vec{\nabla} \times \mu_0 \vec{H})$

or

$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = (-) \mu_0 \frac{d}{dt} (\vec{\nabla} \times \vec{H})$ ------ (6)

From vector calculus identity ,we have

$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} . (\vec{\nabla} . \vec{E}) - \nabla^2 \vec{E}$ ----- (7)

from (5) $(\vec{\nabla} . \vec{E}) = 0$ sub in (7)

$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = - \nabla^2 \vec{E}$ ---------- (8)

sub (8) in (6)

$- \nabla^2 \vec{E} = (-) \ _0 \frac{d}{dt} (\vec{\nabla} \times \vec{H})$

we know that $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{dD}{dt}$

$- \nabla^2 \vec{E} = (-)_0 \frac{d}{dt} (\vec{J} + \frac{dD}{dt})$----------(9)

$- \nabla^2 \vec{E} = (-)_0 \frac{d}{dt} (\frac{\epsilon_0 d\vec{E}}{dt})$

Since $\vec{J} = 0$ and $D = \epsilon_0 \vec{E}$

$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{d^2 \vec{E}}{dt^2}$------------(10)

This is general electromagnetic Wave Equation in terms of electricfield vector($\vec{E}$) for free space.

<u>Wave Equation for electricfield vector($\vec{B}$):</u>

$$(\vec{\nabla} \times \vec{H}) = (\vec{J} + \frac{d\vec{D}}{dt})$$

Taking the curl on both sides of equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times (\vec{J} + \frac{d\vec{D}}{dt})$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \in_0 \frac{d}{dt} \times (\vec{\nabla} \times \vec{E}) \text{ ------------------(11)}$$

From vector calculus identity ,we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} .(\vec{\nabla} . \vec{H}) - \nabla^2 \vec{H} \text{ -----(12)}$$

we know that

$$(\vec{\nabla} . \vec{B}) = 0$$

$$\mu_0(\vec{\nabla} . \vec{H}) = 0$$

(or)

$$(\vec{\nabla} . \vec{H}) = 0 \text{----------------(13)}$$

Sub (13) in (12)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = - \nabla^2 \vec{H} \text{ ----- (14)}$$

Sub (14) in (11)

$$- \nabla^2 \vec{H} = \in_0 \frac{}{dt}(\vec{\nabla} \times \vec{E}) \text{ ---------(15)}$$

Sub (A) in (15)

$$- \nabla^2 \vec{H} = \in_0 \frac{}{dt} (- \frac{d\vec{B}}{dt})$$

$$- \nabla^2 \vec{H} = \in_0 \frac{}{dt} (\mu_0 \vec{H})$$

$$\nabla^2 \vec{H} = \mu_0 \in_0 \frac{d^2 \vec{H}}{t^2} \text{----------------(16)}$$

This is general electromagnetic Wave Equation in terms of electricfield vector($\vec{H}$) for free space.