Problems Using Fourier Transforms Evaluate

(i) $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ (ii) $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$

(iii) $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ (iv) $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

Solution: (i) $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$

Let $f(x) = e^{-ax}$; $g(x) = e^{-bx}$; $a, b > 0$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2} \quad \text{--- (1)}$$

$$F_s[e^{-bx}] = \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+b^2} \quad \text{--- (2)}$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \quad \text{--- (1)}$$

$$F_s[e^{-bx}] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right) \quad \text{--- (2)}$$

Using the property,

$$\int_0^\infty F_s[f(x)]\cdot G_s[g(x)]\,ds = \int_0^\infty f(x)\,g(x)\,dx.$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right)ds$$

$$= \int_0^\infty e^{-ax}\,e^{-bx}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \int_0^\infty f(x)\,g(x)\,dx .$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \int_0^\infty e^{-(a+b)x}\,dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \left(0 - \frac{1}{-(a+b)}\right) \qquad e^{-\infty}=0$$

$$\therefore \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \frac{\pi}{2}\left(\frac{1}{a+b}\right)$$

put $s=x \Rightarrow ds = dx$.

$$\therefore \int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2}\,\frac{1}{(a+b)} .$$

(ii) $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$ Let $f(x)=e^{-ax}$

$$F_S[f(x)] = f_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin(sx)\,dx .$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$$

Using Parseval's Identity,

~~Fourier~~ $$\int_0^\infty [f_s(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx .$$

$$\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\,\frac{s}{a^2+s^2}\right]^2 ds = \int_0^\infty [e^{-ax}]^2\,dx .$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{s^2}{(a^2+s^2)^2}\right) ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds = \frac{\pi}{2}\left[\frac{e^{-\infty}}{-2a} - \frac{e^0}{-2a}\right]$$

put $s=x \Rightarrow ds = dx$. $\qquad = \dfrac{\pi}{2}\left(\dfrac{1}{2a}\right)$

$\therefore \displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$ // $\qquad = \dfrac{\pi}{4a}$.

(iii) $\dfrac{dx}{(x^2+a^2)(x^2+b^2)}$

Let $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

Fourier Cosine Transform,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

Let $g(x) = e^{-bx}$

$$G_c[f(x)] = G_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx}\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{b}{s^2+b^2}\right]$$

Using the Property,

$$\int_0^\infty F_c(s) . G_c(s)\,ds = \int_0^\infty f(x)g(x)\,dx.$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \sqrt{\frac{2}{\pi}} \frac{b}{s^2+b^2}\,ds = \int_0^\infty e^{-ax}e^{-bx}\,dx.$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(s^2+a^2)(s^2+b^2)}\,ds = \int_0^\infty e^{-(a+b)x}\,dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\therefore \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\,ds = \frac{\pi}{2ab}\left[e^{-\infty} - \frac{1}{-(a+b)}\right]$$

$$= \frac{+\pi}{2ab(a+b)}$$

put $s = x$ and $ds = dx$ ∴

$$\therefore \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2ab(a+b)}$$

(iv) $\frac{dx}{(x^2+a^2)^2}$ Let $f(x) = e^{-ax}$

Fourier Cosine Transform,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[s] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right)$$

Using Parseval's Identity,

$$\int_0^\infty [F_c(s)]^2 ds = \int_0^\infty [f(x)]^2 dx$$

$$\int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right) \right]^2 ds = \int_0^\infty [e^{-ax}]^2 dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{a^2}{(s^2+a^2)^2} ds = \left[\frac{e^{-2ax}}{-2a} \right]_0^\infty$$

$$\therefore \int_0^\infty \frac{1}{(s^2+a^2)^2} ds = \frac{\pi}{2a^2} \left[e^{-\infty} - \frac{1}{-2a} \right]$$

$$= \frac{\pi}{2a^2} \left(\frac{1}{2a} \right) = \frac{\pi}{4a^3}.$$

put $s = x$

$ds = dx$.

$$\therefore \int_0^\infty \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}. \; //.$$