1. Find the Fourier Cosine Transform of $\frac{e^{-ax}}{x}$ and hence find $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$.

Solution:

Fourier Cosine Transforms $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(sx)\,dx$

Given: $f(x) = \frac{e^{-ax}}{x}$.

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos(sx)\,dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos(sx)\,dx.$$

Differentiating on both sides w.r.to. 's' we get

$$\frac{d}{ds}F_c(s) = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{ds}\frac{e^{-ax}}{x}\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{d}{ds}\cos(sx)\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\left[-\sin(sx)\,\cancel{x}\right]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}(-\sin(sx))\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin(sx)\,dx$$

$$\frac{d}{ds}F_c(s) = -\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$$

$$\int \frac{d}{ds}F_c(s)\,ds = -\sqrt{\frac{2}{\pi}}\int \frac{s}{s^2+a^2}\,ds$$

$$F_c(s) = -\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int \frac{2s}{s^2+a^2}\,ds$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\log(s^2+a^2)$$

$$= \frac{-1}{\sqrt{2\pi}}\log(s^2+a^2).$$

$$\therefore F_c\left[\frac{e^{-ax}}{x}\right] = \frac{-1}{\sqrt{2\pi}}\log(s^2+a^2). \quad \text{--- (1)}$$

$$\therefore F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}}\log(s^2+b^2) \quad \text{--- (2).}$$

Now,

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{-1}{\sqrt{2\pi}}\log(s^2+a^2) + \frac{1}{\sqrt{2\pi}}\log(s^2+b^2) \quad \text{(from (1) \& (2))}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) \;//.$$

for Fourier Sine Transform:

Use $\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{a}{a^2+b^2}$

and $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c.$

Hint: But $F_s(s)=0$ when $s=0$ $\therefore c=0$.

$$\therefore F_s(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

2) Find the FST and FCT of $xe^{-ax}$

Solution: $F_s[x f(x)] = -\frac{d}{ds}\{F_c[f(x)]\}$ [By known property also.]

Here $f(x) = e^{-ax}$.

$$\therefore F_s[x.e^{-ax}] = -\frac{d}{ds}\{F_c[e^{-ax}]\}$$

We know that $F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$.

$$\therefore F_s[xe^{-ax}] = -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\cdot a\cdot\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)$$

$$= -\sqrt{\frac{2}{\pi}}a\cdot\left[-(s^2+a^2)^{-1-1}(2s)\right]$$

$$= -\sqrt{\frac{2}{\pi}}2as\left[\frac{-1}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{2as}{(s^2+a^2)^2}$$

$$F_s[xe^{ax}] = \sqrt{\frac{2}{\pi}}\frac{2as}{(s^2+a^2)^2} \quad //$$

Similarly,

$$F_c[x f(x)] = \frac{d}{ds}\{F_s[f(x)]\}.$$

$$F_c[xe^{-ax}] = \frac{d}{ds}\{F_s[e^{-ax}]\}.$$

W.K.T, $F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$

$$F_c[xe^{-ax}] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)-s.2s}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$F_c[xe^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] \quad //$$