Deflection of Cantilevers

13.1. INTRODUCTION

Cantilever is a beam whose one end is fixed and other end is free. In this chapter we shall discuss the methods of finding slope and deflection for the cantilevers when they are subjected to various types of loading. The important methods are (*i*) Double integration method (*ii*) Macaulay's method and (*iii*) Moment-area-method. These methods have also been used for finding deflections and slope of the simply supported beams.

13.2. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a point load at the free end B is shown in Fig. 13.1. AB shows the position of cantilever before any load is applied whereas AB' shows the position of the cantilever after loading.

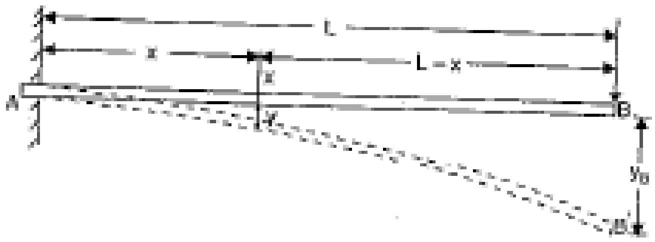

Fig. 13.1

Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -W(L-x) \qquad \text{(Minus sign due to hogging)}$$

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = -W(L-x) = -WL + Wx$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \qquad \ldots(i)$$

Integration again, we get

$$EIy = -WL\frac{x^2}{2} + \frac{W}{2}\frac{x^3}{3} + C_1x + C_2 \quad ...(ii)$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (i) at $x = 0$, $y = 0$ (ii) $x = 0$, $\frac{dy}{dx} = 0$

[At the fixed end, deflection and slopes are zero]

(i) By substituting $x = 0$, $y = 0$ in equation (ii), we get

$$0 = 0 + 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(ii) By substituting $x = 0$, $\frac{dy}{dx} = 0$ in equation (i), we get

$$0 = 0 + 0 + C_1 \quad \therefore \quad C_1 = 0$$

Substituting the value of C_1 in equation (i), we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2}$$

$$= -W\left(Lx - \frac{x^2}{2}\right) \quad ...(iii)$$

The equation (iii) is known as *slope equation*. We can find the slope at any point on the cantilever by substituting the value of x. The slope and deflection are maximum at the free end. These can be determined by substituting $x = L$ in these equations.

Substituting the values of C_1 and C_2 in equation (ii), we get

$$EIy = -WL\frac{x^2}{2} + \frac{Wx^3}{6} \quad (\because \quad C_1 = 0, C_2 = 0)$$

$$= -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \quad ...(iv)$$

The equation (iv) is known as *deflection equation*.

Let θ_B = slope at the free end B i.e., $\left(\frac{dy}{dx}\right)$ at $B = \theta_B$ and

y_B = Deflection at the free end B

(a) Substituting θ_B for $\frac{dy}{dx}$ and $x = L$ in equation (iii), we get

$$EI.\theta_B = -W\left(L.L - \frac{L^2}{2}\right) = -W.\frac{L^2}{2}$$

$$\therefore \quad \theta_B = -\frac{WL^2}{2EI} \quad ...(13.1)$$

Negative sign shows that tangent at B makes an angle in the anti-clockwise direction with AB

$$\therefore \quad \theta_B = \frac{WL^2}{2EI} \quad ...(13.1A)$$

(b) Substituting y_B for y and $x = L$ in equation (iv), we get

$$EI.y_B = -W\left(L.\frac{L^2}{2} - \frac{L^3}{6}\right) = -W\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -W.\frac{L^3}{3}$$

$$\therefore \quad y_B = -\frac{WL^3}{3EI} \quad \text{...(13.2)}$$

(Negative sign shows that deflection is downwards)

$$\therefore \quad \text{Downward deflection, } y_B = \frac{WL^3}{3EI} \quad \text{...(13.2 A)}$$

13.3. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point A and free at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig. 13.2.

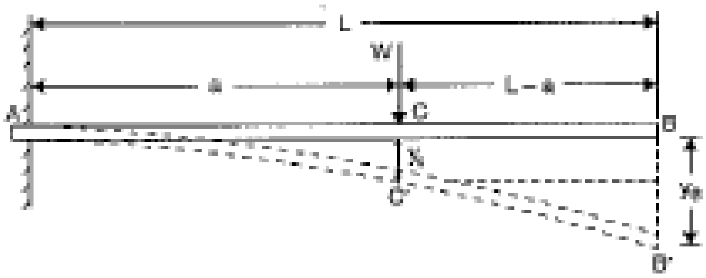

Fig. 13.2

Let θ_C = Slope at point C i.e., $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C

y_B = Deflection at point B

The portion AC of the cantilever may be taken as similar to a cantilever in Art. 13.1 (i.e., load at the free end).

$$\therefore \quad \theta_C = +\frac{Wa^2}{2EI} \quad \text{[In equation (13.1 A) change } L \text{ to } a]$$

and

$$y_C = \frac{Wa^3}{3EI} \quad \text{[In equation (13.2 A) change } L \text{ to } a]$$

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore

Slope at C = slope at B

or

$$\theta_C = \theta_B = \frac{Wa^2}{2EI} \quad \text{...(13.3)}$$

Now from Fig. 13.2, we have

$$y_B = y_C + \theta_C(L - a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a) \quad \left(\because \theta_C = \frac{Wa^2}{2EI}\right) \quad \text{...(13.4)}$$

Problem 13.1. *A cantilever of length 3 m is carrying a point load of 25 kN at the free end. If the moment of inertia of the beam = 10^8 mm^4 and value of $E = 2.1 \times 10^5$ N/mm^2, find (i) slope of the cantilever at the free end and (ii) deflection at the free end.*

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

Point load, $W = 25$ kN $= 25000$ N

M.O.I., $I = 10^8$ mm^4

Value of $E = 2.1 \times 10^5$ N/mm^2

(*i*) Slope at the free end is given by equation (13.1 A).

$$\therefore \quad \theta_B = \frac{WL^2}{2EI} = \frac{25000 \times 3000^2}{2 \times 2.1 \times 10^5 \times 10^8} = \mathbf{0.005357\ rad.} \quad \textbf{Ans.}$$

(*ii*) Deflection at the free end is given by equation (13.2 A),

$$y_B = \frac{WL^3}{3EI} = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8} = \mathbf{10.71\ mm.} \quad \textbf{Ans.}$$

Problem 13.2. *A cantilever of length 3 m is carrying a point load of 50 kN at a distance of 2 m from the fixed end. If $I = 10^8$ mm^4 and $E = 2 \times 10^5$ N/mm^2, find (i) slope at the free end and (ii) deflection at the free end.*

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

Point load, $W = 50$ kN $= 50000$ N

Distance between the load and the fixed end,

$a = 2$ m $= 2000$ mm

M.O.I., $I = 10^8$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

(*i*) Slope at the free end is given by equation (13.3) as

$$\theta_B = \frac{Wa^2}{2EI} = \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8} = \mathbf{0.005\ rad.} \quad \textbf{Ans.}$$

(*ii*) Deflection at the free end is given by equation (13.4) as

$$y_B = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a)$$

$$= \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000 - 2000)$$

$$= 6.67 + 5.0 = \mathbf{11.67\ mm.} \quad \textbf{Ans.}$$

13.4. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w per unit length over the whole length, is shown in Fig. 13.3.

Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -w(L-x) \cdot \frac{(L-x)}{2} \qquad \text{(Minus sign due to hogging)}$$

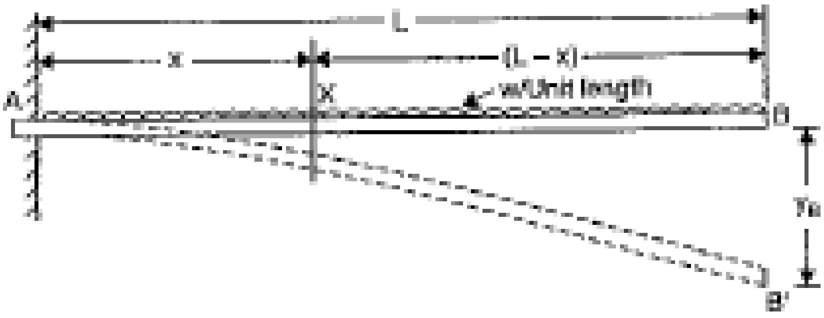

Fig. 13.3

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -\frac{w}{2}\frac{(L-x)^3}{3}(-1) + C_1$$

$$= \frac{w}{6}(L-x)^3 + C_1 \qquad \text{...}(i)$$

Integrating again, we get

$$EIy = \frac{w}{6}\cdot\frac{(L-x)^4}{4}(-1) + C_1x + C_2$$

$$= -\frac{w}{24}(L-x)^4 + C_1x + C_2 \qquad \text{...}(ii)$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0$, $y = 0$ and (*ii*) at $x = 0$, $\frac{dy}{dx} = 0$ (as the deflection and slope at fixed end A are zero).

(*i*) By substituting $x = 0$, $y = 0$ in equation (*ii*), we get

$$0 = -\frac{w}{24}(L-0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$$

$$\therefore \qquad C_2 = \frac{wL^4}{24}$$

(*ii*) By substituting $x = 0$ and $\frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = \frac{w}{6}(L-0)^3 + C_1 = \frac{wL^3}{6} + C_1$$

$$\therefore \qquad C_1 = -\frac{wL^3}{6}$$

Substituting the values of C_1 and C_2 in equation (*i*) and (*ii*), we get

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \quad \text{...(iii)}$$

and

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \quad \text{...(iv)}$$

The equation (*iii*) is known as *slope equation* and equation (*iv*) as *deflection equation*. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point B, the value of $x = L$ is substituted in these equations.

Let θ_B = Slope at the free end B *i.e.*, $\left(\frac{dy}{dx}\right)$ at B

y_B = Deflection at the free end B.

From equation (*iii*), we get slope at B as

$$EI.\theta_B = \frac{w}{6}(L-L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$$

$$\therefore \quad \theta_B = -\frac{wL^3}{6EI} = -\frac{WL^2}{6EI} \quad (\because W = \text{Total load} = w.L) \text{ ...(13.5)}$$

From equation (*iv*), we get the deflection at B as

$$EI.y_B = -\frac{w}{24}(L-L)^4 - \frac{wL^3}{6} \times L + \frac{wL^4}{24}$$

$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$$

$$\therefore \quad y_B = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI} \quad (\because W = w.L)$$

$\therefore$ Downward deflection at B,

$$y_B = \frac{wL^4}{8EI} = \frac{WL^3}{8EI} \quad \text{...(13.6)}$$

Problem 13.3. *A cantilever of length 2.5 m carries a uniformly distributed load of 16.4 kN per metre length over the entire length. If the moment of inertia of the beam = 7.95 × 10^7 mm^4 and value of E = 2 × 10^5 N/mm^2, determine the deflection at the free end.*

Sol. Given :

Length, L = 2.5 mm = 2500 mm

U.d.l., w = 16.4 kN/m

$\therefore$ Total load, $W = w \times L$ = 16.4 × 2.5 = 41 kN = 41000 N

Value of $I = 7.95 \times 10^7$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

Let y_B = Deflection at the free end.

Using equation (13.6), we get

$$y_B = \frac{WL^3}{8EI} = \frac{41000 \times 2500^3}{8 \times 2 \times 10^5 \times 7.95 \times 10^7}$$

= **5.036 mm. Ans.**

Problem 13.7. *A cantilever of length 3 m carries two point loads of 2 kN at the free end and 4 kN at a distance of 1 m from the free end. Find the deflection at the free end.*

Take $E = 2 \times 10^5$ N/mm² and $I = 10^8$ mm⁴.

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

Load at free end, $W_1 = 2$ kN $= 2000$ N

Load at a distance one m from free end,

$W_2 = 4$ kN $= 4000$ N

Distance AC, $a = 2$ m $= 2000$ mm

Value of $E = 2 \times 10^5$ N/mm²

Value of $I = 10^8$ mm⁴

Let y_1 = Deflection at the free end due to load 2 kN alone

y_2 = Deflection at the free end due to load 4 kN alone.

Fig. 13.6

Downward deflection due to load 2 kN alone at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{2000 \times 3000^3}{3 \times 2 \times 10^5 \times 10^8} = 0.9 \text{ mm.}$$

Downward deflection at the free end due to load 4 kN (*i.e.*, 4000 N) alone at a distance 2 m from fixed end is given by (13.4) as

$$y_2 = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a)$$
$$= \frac{4000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{4000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000 - 2000)$$
$$= 0.54 + 0.40 = 0.94 \text{ mm}$$

∴ Total deflection at the free end

$$= y_1 + y_2$$
$$= 0.9 + 0.94 = \mathbf{1.84 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 13.8. *A cantilever of length 2 m carries a uniformly distributed load of 2.5 kN/m run for a length of 1.25 m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and E = 1 × 10⁴ N/mm².* (Annamalai University, 1990)

Sol. Given :

Length, $L = 2 \text{ m} = 2000 \text{ mm}$

U.d.l., $w = 2.5 \text{ kN/m} = 2.5 \times 1000 \text{ N/m}$

$$= \frac{2.5 \times 1000}{1000} \text{ N/mm} = 2.5 \text{ N/mm}$$

Point load at free end, $W = 1 \text{ kN} = 1000 \text{ N}$

Distance AC, $a = 1.25 \text{ m} = 1250 \text{ mm}$

Width, $b = 12 \text{ mm}$

Depth, $d = 24 \text{ mm}$

Value of $I = \frac{bd^3}{12} = \frac{12 \times 24^3}{12}$

$$= 13824 \text{ cm}^4 = 13824 \times 10^4 \text{ mm}^4 = 1.3824 \times 10^8 \text{ mm}^4$$

Value of $E = 1 \times 10^4 \text{ N/mm}^2$

Let y_1 = Deflection at the free end due to point load 1 kN alone

y_2 = Deflection at the free end due to u.d.l. on length AC.

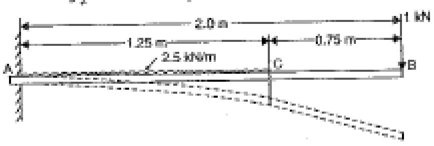

Fig. 13.7

(*i*) Now the downward deflection at the free end due to point load of 1 kN (or 1000 N) at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{1000 \times 2000^3}{3 \times 10^4 \times 1.3824 \times 10^8} = 1.929 \text{ mm.}$$

(*ii*) The downward deflection at the free end due to uniformly distributed load of 2.5 N/mm on a length of 1.25 m (or 1250 mm) is given by equation (13.8) as

$$y_2 = \frac{wa^4}{8EI} + \frac{w \cdot a^3}{6EI}(L-a)$$

$$= \frac{2.5 \times 1250^4}{8 \times 10^4 \times 1.3824 \times 10^8} + \frac{2.5 \times 1250^3}{6 \times 10^4 \times 1.3824 \times 10^8}(2000 - 1250)$$

$$= 0.5519 + 0.4415 = 0.9934$$

∴ Total deflection at the free end due to point load and u.d.l.

$$= y_1 + y_2 = 1.929 + 0.9934 = \mathbf{2.9224\ mm.}\quad \textbf{Ans.}$$

Problem 13.9. *A cantilever of length 2 m carries a uniformly distributed load 2 kN/m over a length of 1 m from the free end, and a point load of 1 kN at the free end. Find the slope and deflection at the free end if $E = 2.1 \times 10^5$ N/mm² and $I = 6.667 \times 10^7$ mm⁴.*

Sol. Given : (See Fig. 13.8)

Length, $L = 2$ m $= 2000$ mm

U.d.l. $w = 2$ kN/m $= \frac{2 \times 1000}{1000}$ N/mm $= 2$ N/mm

Length *BC*, $a = 1$ m $= 1000$ mm

Point load, $W = 1$ kN $= 1000$ N

Value of $E = 2.1 \times 10^5$ N/mm²

Value of $I = 6.667 \times 10^7$ mm⁴.

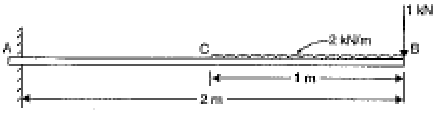

Fig. 13.8

(i) Slope at the free end

Let θ_1 = Slope at the free end due to point load of 1 kN *i.e.*, 1000 N

θ_2 = Slope at the free end due to u.d.l. on length *BC*.

The slope at the free end due to a point load of 1000 N at *B* is given by equation (13.1 A) as

$$\theta_1 = \frac{WL^2}{2EI} \qquad (\because \ \theta_B = \theta_1 \text{ here})$$

$$= \frac{1000 \times 2000^2}{2 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.0001428 \text{ rad.}$$

The slope at the free end due to u.d.l. of 2 kN/m over a length of 1 m from the free end is given by equation (13.9) as

$$\theta_2 = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI} \qquad (\because \ \theta_B = \theta_2 \text{ here})$$

$$= \frac{2 \times 2000^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000 - 1000)^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}$$

$$= 0.0001904 - 0.000238 = 0.0001666 \text{ rad.}$$

∴ Total slope at the free end

$$= \theta_1 + \theta_2 = 0.0001428 + 0.0001666 = \mathbf{0.0003094\ rad.}\quad \textbf{Ans.}$$

(ii) Deflection at the free end

Let y_1 = Deflection at the free end due to point load of 1000 N

y_2 = Deflection at the free end due to u.d.l. on length *BC*.

The deflection at the free end due to point load of 1000 N is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} \qquad (\because \ \text{Here } y_1 = y_B)$$

$$= \frac{1000 \times 2000^3}{3 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.1904 \text{ mm.}$$

The deflection at the free end due to u.d.l. of 2 N/mm over a length of 1 m from the free end is given by equation (13.10) as

$$y_2 = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]$$

$$= \frac{2 \times 2000^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \left[\frac{2(2000 - 1000)^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} + \frac{2(2000 - 1000)^3 \times 1000}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}\right]$$

$$= 0.2857 - [0.01785 + 0.0238] = 0.244 \text{ mm}$$

∴ Total deflection at the free end

$$= y_1 + y_2 = 0.1904 + 0.244 = \mathbf{0.4344\ mm.}\quad \textbf{Ans.}$$