MACAULAY'S METHOD

The procedure of finding slope and deflection for a simply supported beam with an eccentric point load is a very laborious. There is a convenient method for determining the deflection of the beam subjected to point loads.

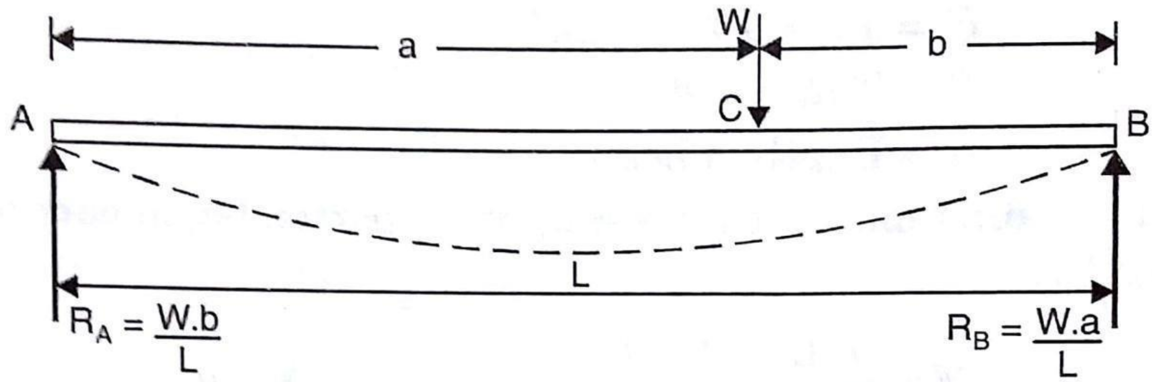
This method was devised by Mr. M.H. Macaulay and is known as Macaulay's method. This method mainly consists in a special manner in which the bending moment at any section is expressed and in the manner in which the integrations are carried out.

3.3.1. DEFLECTION OF A SIMPLY SUPPORTED LOAD WITH AN ECCENTRIC POINT LOAD

A simply supported beam AB of length L and carrying a point load W at a distance 'a' from left support and at a distance 'b' from right support is shown in Fig. The reaction at A and B are given by,

$R_A = \frac{Wb}{L}$ and $R_B = \frac{Wa}{L}$

The bending moment at any section between A and C at a distance x from A is given by, $M_x = R_A \times x = \frac{Wb}{L} \times x$



The above equation of B.M. holds good for the values of x between 0 and 'a'. The bending moment at any section between C and B at a distance x from A is given by,

$M_x = R_A \times x - W \times (x-a) = \frac{Wb}{L} \times x - W(x-a)$

The above equation of B.M holds good for all values of x between x = a and x = b.

The B.M for all sections of the beam can be expressed in a single equation written as

$M_x = \frac{Wb}{L}$ X x ⋮ – W (x-a) … (i)

Stop at the dotted line for any point in section AC. But for any point in section CB, add the expression beyond the dotted line also. The B.M. at any section is also given by

$M = EI \frac{d^2y}{dx^2}$…(ii)

Hence equating (i) and (ii), we get

$EI \frac{d^2y}{dx^2} = \frac{Wb}{L}$ X x ⋮ – W (x-a) …(iii) Integrating the above equation, we get

$EI\frac{dy}{dx} = \frac{Wb}{L}\frac{x^2}{2} + C_1 \;\vdots\; - \frac{W\,(x-a)^2}{2}$ …(iv)

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole. Hence the integration of (x-a) will be $\frac{(x-a)^2}{2}$ and not $\frac{x^2}{2}$ - ax.

Integrating equation (iv) once again, we get

$EIy = \frac{Wb}{2L}\frac{x^3}{3} + C_1x + C_2 \;\vdots\; -\frac{W}{2}\frac{(x-a)^3}{3}$ …(v)

Where C_2 is another constant of integration. This constant is written after C_1x. The integration of $(x-a)^2$ will be $\frac{(x-a)^3}{3}$. This type of integration is justified as the constants of integrations C_1 and C_2 are valid for all values of x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = L, y = 0.since deflection is 0 at A and B)

(i) At A, x =0 and y = 0. Substituting these values in equation (v) upto dotted line only as the point A lies in AC (i.e. at first portion), we get

$0 = 0+0+C_2$

$\therefore \quad C_2 = 0$

(ii).At B, x = L and y = 0. Substituting these values in equation (v), we get

$0 = \frac{Wb}{2L}\frac{L^3}{3} + C_1L + 0 - \frac{W}{2}\frac{(L-a)^3}{3}$

$= \frac{WbL^2}{6} + C_1L - \frac{Wb^3}{6}$ (since L-a = b)

$\therefore\ C_1 L = \frac{Wb^3}{6} - \frac{WbL^2}{6} = -\frac{Wb(L^2-b^2)}{6}$

$\therefore\ C_1 = -\frac{Wb(L^2-b^2)}{6L}$ …(vi)

Substituting the value of C_1 in equation (iv), we get

$$EI\frac{dy}{dx} = \frac{Wb}{L}\frac{x^2}{2} - \frac{Wb(L^2-b^2)}{6L} \ \vdots\ -\frac{W(x-a)^2}{2} \quad \ldots(vii)$$

Equation (vii) gives the slope at any point in the beam. Slope is maximum at A or B. To find the slope at A, substitute x = 0 in the above equation upto dotted line as point A lies in AC.

$EI\theta A = \frac{Wb}{2L} X\ 0 - \frac{Wb(L^2-b^2)}{6L}$

$= -\frac{Wb(L^2-b^2)}{6L}$

$\therefore \theta_A = -\frac{Wb(L^2-b^2)}{6EIL}$

Substituting the values of C_1 and C_2 in equation (v), we get

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$$EIy = \frac{Wb}{2L}\frac{x^3}{3} - \frac{Wb(L^2-b^2)}{6L}x + 0 \ \vdots\ -\frac{W}{2}\frac{(x-a)}{3} \quad \ldots(Viii)$$

Equation (viii) gives the deflection at any point in the beam. To find the deflection y_c under the load, substitute x = a in equation (viii) and consider the equation upto dotted line (as point C lies in AC). Hence, we get

$EIy_c = \frac{Wb}{2L}\frac{a^3}{3} - \frac{Wb(L^2-b^2)a}{6L}$

$= \frac{Wb}{6L}.a.(a^2 - L^2 + b^2)$

$= -\frac{Wab}{6L}(L^2-a^2-b^2)$

$= \frac{Wab}{6L}\left[(a+b)^2 - a^2 - b^2\right]$ (since L = a+b)

$= -\frac{Wab}{6L}(2ab)$

$= -\frac{Wa^2b^2}{3L}$

$\therefore\ y_c = -\frac{Wa^2b^2}{3EIL}$

Example.3.3.1. A horizontal beam of uniform section and 6 meters long is simply supported at its ends. Two vertical concentrated loads of 48 kN and 40 kN act at 1m and 3m respectively from the left hand support. Determine the magnitude of the deflection under the loads and maximum deflection using Macaulay's method. If E = 200GN/m^2 and I = 85 X 10^{-6} m^4

Given Data

W_C = 48 kN

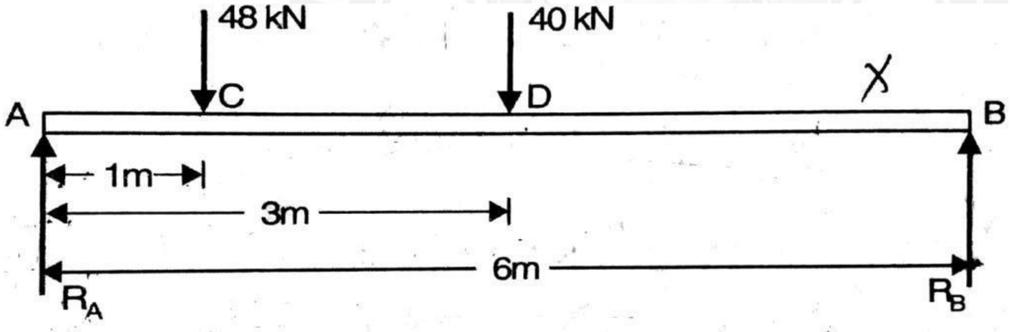
W_D = 40 kN

E = 200GN/m^2 = 200 X 106kN/m^2

I = 85 X 10^{-6} m^4

To Find:

The deflection under the loads and the maximum deflection **Solution:**



Taking moment about A,

R_B X 6 – (40 X 3) –(48 X 1) = 0

6R_B – 120 -48 = 0

6R_B = 168

$R_B = \frac{168}{6} = 28$ KN

R_A +R_B =48+ 40

R_A + 28 = 88

R_A = 88-28 =60 kN

BM for the section X-X

$M_x = R_A \text{ X } x \,\vdots - 48(x-1) \,\vdots -40(x-3)$

$= 60\text{X } x \,\vdots - 48(x-1) \,\vdots -40(x-3)$ … (i)

The B.M. at any section is also given by

$M = EI \frac{d^2y}{dx^2}$…(ii)

Hence equating (i) and (ii), we get

$EI \frac{d^2y}{dx^2} = 60X\ x \vdots - 48(x-1) \vdots -40(x-3)$ …(iii) Integrating the above equation, we get

$\frac{dy}{} = 60 \frac{x^2}{} + C_1 \vdots - \frac{48(x-1)^2}{} \vdots - \frac{40(x-3)}{}$ 2

EI…(iv)

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole. Integrating equation (iv) once again, we get

$EIy = \frac{x^3}{} + C_1x + C_2 \vdots - \frac{48(x-1)}{} \vdots - \frac{40(x-3)}{}$ 3 3

60…(v)

Where C_2 is another constant of integration. This constant is written after C_1x. The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = 6m, y = 0. (since deflection is 0 at A and B) (ii) At A, x =0 and y = 0. Substituting these values in equation (v) uptothe first dotted line only as the point A lies in AC (i.e. at first portion), we get

$0 = 0+0+C_2$

$\therefore C_2 = 0$

(ii).At B, x = 6m and y = 0. Substituting these values in equation (v), we get

$0 = 60 \frac{6^3}{3} + 6C_1 + 0 - \frac{48(6-1)}{3} - \frac{40(6-3)}{3}$ 3 3 X

$6C_1 = -980$

$C_1 = -\frac{980}{6} = -163.33$…(vi)

Substituting the value of C_1 in equation (iv), we get

$\frac{dy}{dx} = \frac{60x}{2} - 163.33 \vdots - \frac{48\ (x-1)}{2} \vdots - \frac{40\ (x-3)}{6}$ 2 2 2

EI…(vii)

Equation (vii) gives the slope at any point in the beam.

Substituting the values of C_1 and C_2 in equation (v), we get

EIy= $60\frac{x^3}{6} - 163.33x \ \vdots - \frac{48(x-1)^3}{6} \ \vdots - \frac{40(x-3)^3}{6}$ …(viii)

Deflection at the point C.

This is obtained by substituting x =1 in equation (viii) up to the first dotted line we get,

$EIy_c = 10 \times 1^3 - 163.33 \times 1$

$= -153.33$ kNm^3

$\therefore y_c = \frac{-153.33}{EI} = \frac{-153.33}{200X10^6 85X10^{-6}}$

$= -0.00902m$

= -9.02mm

Deflection at the point D.

This is obtained by substituting x =3 in equation (viii) up to the Second dotted line we get,

$EIy_D = 10 \times 3^3 - 163.33 \times 3 - 8(3-1)^3$

$= -283.99$ kNm^3

$\therefore yc = \frac{-283.99}{EI} = \frac{-283.99}{200X10^6 85X10^{-6}} = -0.0167m$

= - 16.7 mm

Maximum Deflection

The maximum deflection should be between C and D Where $\frac{dy}{dx} = 0$.

$\therefore$ Put $\frac{dy}{dx} = 0$. In equation (vii) up to second dotted line only

$\frac{60x^2}{2} - 163.33 - \frac{48\,(x-1)^2}{2} = 0$

Or $30x^2 - 163.33 - 24(x^2 - 2x + 1) = 0$

Or $30x^2 - 163.33 - 24x^2 + 48x - 24 = 0$

By solving We get,

X = 2.87m or x = -10.87m

X = -10.87m is not possible , so we take

X = 2.87mWhen x = 2.87m, $y = y_{max}$

Substituting the above condition in equation viii up to the second dotted line we get,

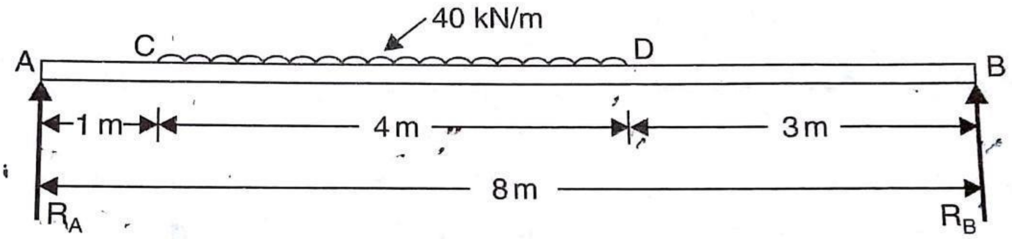
$EIy^{max} = \frac{60 \ X \ 2.87^3}{6} - (163.33 \ X \ 2.87) - \frac{48(2.87-1)^3}{6}$

$EIy_{max} = 236.399 - 468.757 - 52.31 = -284.668$

$\therefore y_{max} = \frac{-284.668}{EI} = \frac{-284.668}{200X10^6 85X10^{-6}} = -.01674m$

$\therefore$ **y_{max} = -16.74 mm**

Example.3.3.2. A beam of length 8m is simply supported at its ends. It carries a uniformly distributed load of 40 kN/m as shown in Fig. Determine the deflection of the beam at its mid point and also the position of maximum deflection and maximum deflection. Take E = 2 X 10^5N/mm^2 and I = 4.3 X 10^8mm



Given Data:

Length, L = 8m

u.d.l, w = 40KN/m

E = 2 X 10^5N/mm^2

I = 4.3 X 10^8mm^4

To find

(i) The central deflection

(ii) The position and magnitude of maximum deflection. **Solution**

Taking moment about A,

R_B X 8 – 40 X 4 X $\left[1 + \frac{4}{2}\right] = 0$

$8R_B - 480 = 0$

$8R_B = 480$

$\therefore \quad R_B = \frac{480}{8} = 60$ KN

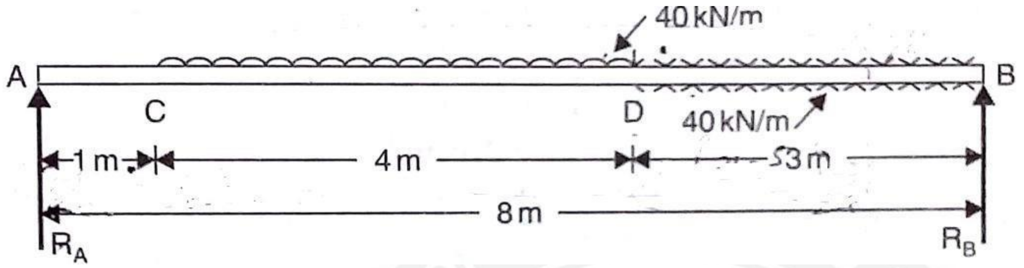
$R_A + R_B = (40 \text{ x } 4)$

$\therefore \quad R_A + 60 = 160$

$\therefore R_A = 160 - 60 = 100$ kN

In ordet to obtain the general expression for the bending moment at a distance x from the left end A, which will apply for all values of x, it is necessary to extend the udlupto the support B, compensating with an equal upward load of 40kN/m over the span DB as shown in Fig.

Now Macaulay's method can be applied.



BM for the section X-X is given by,

$M_x = R_A \text{ X } x \vdots - 40\,(x-1) \text{ X } \frac{(x-1)}{2} \vdots + 40\,(x-5) \text{ X } \frac{(x-5)}{2}$

$= 100 \text{X } x \vdots - 20(x-1)^2 \vdots + 20(x-5)^2$

The B.M. at any section is also given by

$M = EI\,\frac{d^2y}{dx^2}$

Equating the both values of B.M we get,

$EI\,\frac{d^2y}{dx^2} = 100 \text{X } x \vdots - 20(x-1)^2 \vdots + 20(x-5)^2$ Integrating the above equation, we get

$\frac{dy}{dx} = 100\frac{x^2}{2} + C_1 \vdots - \frac{20(x-1)^3}{3} \vdots + \frac{20(x-5)}{3}$ 3

EI …(i)

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole. Integrating the above equation once again, we get,

$EIy = 50\,\frac{x^3}{3} + C_1x + C_2 \vdots - \frac{20(x-1)^4}{3 \text{ X } 4} \vdots + \frac{20(x-5)^4}{3 \text{ X } 4}$

$$= 50\frac{x^3}{3} + C_1x + C_2 \vdots - \frac{5(x-1)^4}{3} \vdots + \frac{5(x-5)^4}{3} \quad \ldots \text{(ii)}$$

Where C_2 is another constant of integration. This constant is written after C_1x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = 8m, y = 0. (since deflection is 0 at A and B) (i) At A, x =0 and y = 0. Substituting these values in equation (ii) uptothe first dotted line only as the point A lies in AC (i.e. at first portion), we get

$0 = 0+0+C_2$

$\therefore C_2 = 0$

(ii).At B, x = 8m and y = 0. Substituting these values in equation (ii), we get

$0 = \frac{50}{3} X\ 8^3 + 8C_1\ +\ 0\ - \frac{5(8-1)^4}{3} + \frac{5(8-5)^4}{3}$

$8C_1 = -4666.67$

$C_1 = -\ \frac{4666.67}{8} = -583.33$

Substituting the value of C_1 in equation (ii), we get

EIy= $50\frac{x^3}{3} - 583.33x\ \vdots - \frac{5(x-1)^4}{3} \vdots + \frac{5(x-4)^4}{3}$ …(iii)

Deflection at the Centre.

This is obtained by substituting x =4 in equation (iii) up to the second dotted line we get,

$$EIy = 50\,\frac{4^3}{3} - 583.33\ X\ 4\ - \frac{5(4-1)^4}{3}$$

$= -\ 1401.66\ kNm^3$

$= -\ 1401.66\ X\ 10^{12}\ Nmm^3$

$\therefore y = \frac{-1401.66\ X 10^{12}}{EI} = \frac{-1401.66\ X\ 10^{12}}{2\ X\ 10^5\ X\ 4.3\ X\ 10^8}$

= -16.3mm

(i) Maximum Deflection

The maximum deflection should be between C and D Where $\frac{dy}{dx} = 0.$

$\therefore$ Put $\frac{dy}{dx}$ = 0. In equation (i) up to second dotted line only

$0 = 100\frac{x^2}{2} + C_1 - \frac{20(x-1)^3}{3}$

$0 = 50x^2 - 583.33 - 6.667(x-1)^3$ …(iv)

The above equation is solved by trial and error method.

Let x=1, then R.H.S of equation (iv)

= 50-583.33-6.667 X 0 = -533.33

Let x=2, then R.H.S = 50 X 4 – 583.33 -6.667 X 1 = -390

Let x=3, then R.H.S = 50 X 9 – 583.33 -6.667 X 8 = -136.69

Let x=4, then R.H.S = 50 X 16 – 583.33 -6.667 X 27 = +36.58

In equation (iv), when x = 3 then R.H.S is negative but when x =4 then R.H.S is positive.

Hence exact value of x lies between 3 and 4

Let x=3.82, then R.H.S= 50 X 3.82 – 583.33 – 6.667$(3.82-1)^3$= -3.22

Let x=3.83, then R.H.S= 50 X 3.83 – 583.33 – 6.667$(3.83-1)^3$ = -0.99

The R.H.S is approximately zero

∴ x = 3.83m.

Hence maximum deflection will be at a distance of 3.83m from support A.

When x = 3.83m, y = y_{max}

Substituting the above condition in equation viii up to the second dotted line we get,

$$EI y^{max} = \frac{50\ X\ 3.83^3}{3} - (583.33\ X\ 3.83) - \frac{5(3.83-1)^4}{3}$$

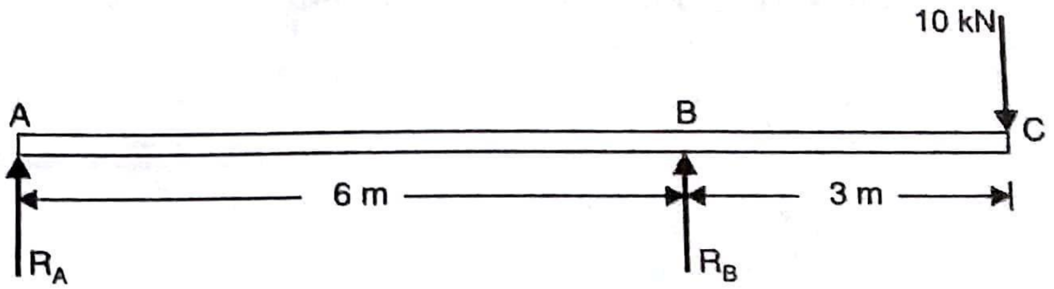
EIy_{max}= -1404.69 kNm^3 = - 1404.69 X $10^{12}Nmm^3$

$$\therefore y_{max} = \frac{-1404.69\ X\ 10^{12}}{EI} = \frac{-1404.69\ X\ 10^{12}}{2\ X\ 10^5\ X\ 4.3\ X\ 10^8}$$

∴y_{max} = -16.33 mm

Example.3.3.3. An overhanging beam ABC is loaded as shown in Fig. Find the slopes over each support and at the right end. Find also the maximum upward deflection between supports and the deflection at the right end.

Take E = 2 X $10^5 N/mm^2$ and I = 5 X $10^8 mm^4$.



Sol.Given:

Point load, W = 10 KN E = 2 X 10^5 N/mm^2.

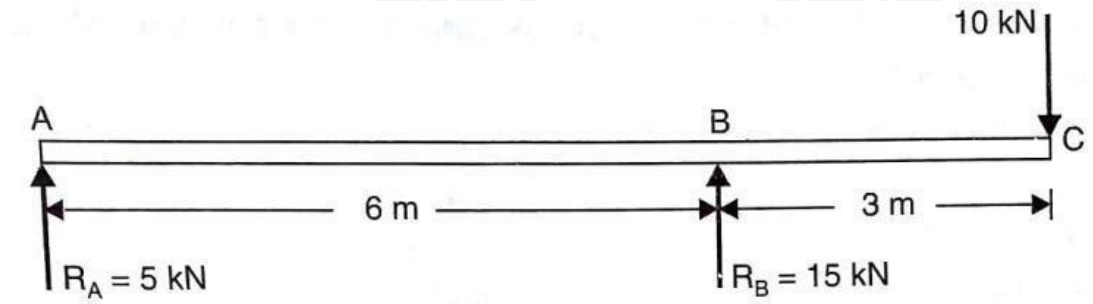
I = 5 X 10^8 mm^4

Taking moments about A, we get R_B X 6 = 10 X 9

$\therefore R_B = \frac{10 X 9}{6} = 15$ kN

$\therefore R_A$ = Total load - R_B = 10-15 = -5 kN

Hence the reactions R_A will be in the downward direction. Hence above Fig. will be modified as shown in following fig. Now write down an expression for the B.M in the last section of the beam.



BM for the section X-X is given by,

$$EI\frac{d^2y}{dx^2} = R_A \text{ X } x \vdots + R_B (x-6)$$

$$= -5x \vdots + 15 (x-6)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -5\frac{x^2}{2} + C_1 \vdots + \frac{15(x-6)^2}{2} \quad \ldots(i)$$

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole. Integrating the above equation once again, we get,

$$EIy = -\frac{5}{2}\frac{x^3}{3} + C_1x + C_2 \vdots + \frac{15(x-6)^3}{2 X 3}$$

$$= -5\frac{x^3}{6} + C_1x + C_2 \vdots + \frac{5(x-6)^3}{2} \quad \ldots (ii)$$

Where C_2 is another constant of integration. This constant is written after C_1x. The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = 6m, y = 0. (since deflection is 0 at A and B) (ii) At A, x =0 and y = 0. Substituting these values in equation (ii) uptothe dotted line only as the point A lies in AB (i.e. at first portion), we get

$0 = 0+0+C_2$

$\therefore C_2 = 0$

(ii).At B, x = 6m and y = 0. Substituting these values in equation (ii), we get

$0 = -\frac{5}{6} X\ 6^3 + 6C_1 + 0$

$6C_1 = 5 X 36$

$C_1 = \frac{5 X 36}{8} = 30$

Substituting the value of C_1 and C_2 in equation (i) & (ii), we get,

$EI\frac{dy}{dx} = -5\frac{x^2}{2} + 30 \ \vdots + \frac{15(x-6)^2}{2}$ …(iii)

And$EI_y = -5\frac{x^3}{6} + 30x + 0 \ \vdots + \frac{5(x-6)^3}{2}$ …(vi)

slope over the support A

By substituting x= 0 in equation (iii) upto dotted line, we get the slope at Support A (the point x = 0 lies in the first part AB of the beam)

$\therefore EI\theta_A = -\frac{5}{2} X\ 0 + 30 = 30 kNm^2 = 30 X 10^9 Nmm^2$

$\theta_A = \frac{30 X 10^9}{EI} = \frac{30 X 10^9}{2X\ 10^5 X 5 \times 10^8}$ = **0.0003 radians.**

Slope at the support B

By substituting x= 6 m in equation (iii) upto dotted line, we get the slope at Support B (the point x = 6 lies in the first part AB of the beam)

$\therefore EI\theta_B = -\frac{5}{2} X\ 6^2 + 30 = -60 kNm^2 = -60 X 10^9 Nmm^2$

$\theta_B = \frac{-60 X 10^9}{E \times I} = \frac{-60 X 10^9}{2X\ 10^5 X\ 5 \times 10^8}$

= - 0.0006 radians.

Slope at the right end i.e., at C

By substituting x= 9 m in equation (ii), we get the slope at C. In this case, complete equation is to be taken as the point x = 9 m lies in the last part of the beam)

$\therefore EI\theta_C = -\frac{5}{2} X\ 9^2 + 30 + \frac{15}{2}(9 - 6)^2 = -105 kNm^2$

$= -105 X 10^9 Nmm^2$

$\theta_C = \frac{-105 \text{ X } 10^9}{E \times I} = \frac{-105 \text{ X } 10^9}{2X\ 10^5 X5 \times 10^8}$

= **- 0.00105 radians**.

Maximum upward deflection between the supports

For the maximum deflection between the supports, $\frac{dy}{dx}$ should be zero. Hence equating the slope given by equation (iii) to be zero upto dotted line, we get

$0 = -\frac{5}{2}x^2 + 30 = -5x^2 + 60$

or $5x^2 = 60$ or $x = \sqrt{\frac{60}{5}} = \sqrt{12}$ = 3.464 m

Now substituting x = 3.464 m in equation (iv) upto dotted line, we get maximum deflection as

$EIy^{max} = -\frac{5}{6} \times 3.464^3 + 30 \times 3.464$

$= 69.282 \text{ kNm}^3$

$= 69.282 \times 1000 \times 10^9 \text{ Nmm}^3 = 69.282 \times 10^{12} \text{ mm}^3$

$y^{max} = \frac{69.282 \times 10^{12}}{2 \times 10^5 \times 5 \times 10^8}$

= 0.6928 mm(upward)

Deflection at the right end i.e., at point C

By substituting x= 9 m in equation (iv), we get the deflection at point C. Here complete equation is to be taken as the point x = 9 m lies in the last part of the beam.

$EIy_c = -\frac{5}{6} \times 9^3 + 30 \times 9 + \frac{5}{2}(9-6)^3$

$= 270 \text{ kNm}^3 = -270 \times 10^{12} \text{Nmm}^3$

$y_c = \frac{-270 \times 10^{12}}{2 \times 10^5 \times 5 \times 10^8}$

= **- 2.7mm(downward)**