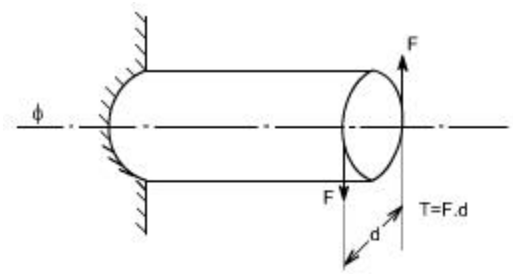
UNIT III TORSION

carriage springs.

Torsion of circular shafts

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross � section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.

GENERATION OF SHEAR STRESSES

The physical understanding of the phenomena of setting up of shear stresses in a shaft subjected to a torsion may be understood from the figure 1-3.

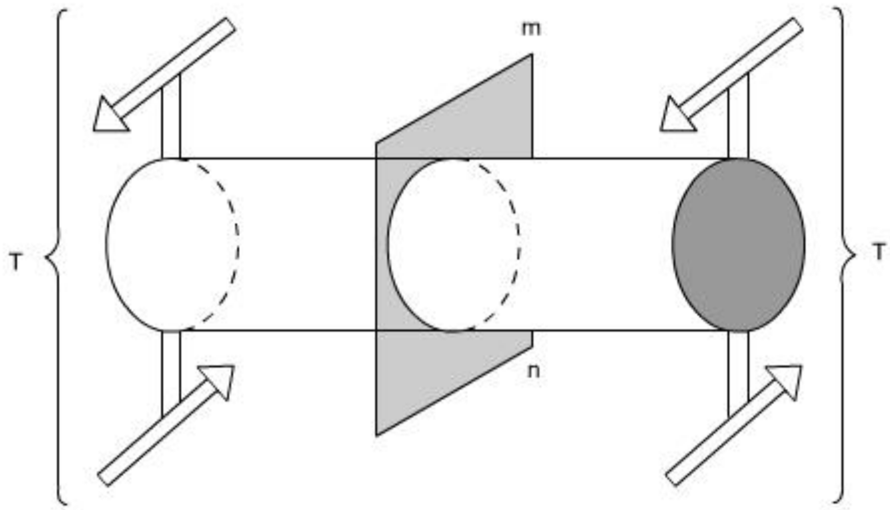


Fig 1: Here the cylindrical member or a shaft is in static equilibrium where T is the resultant external torque acting on the member. Let the member be imagined to be cut by some imaginary plane �mn'.

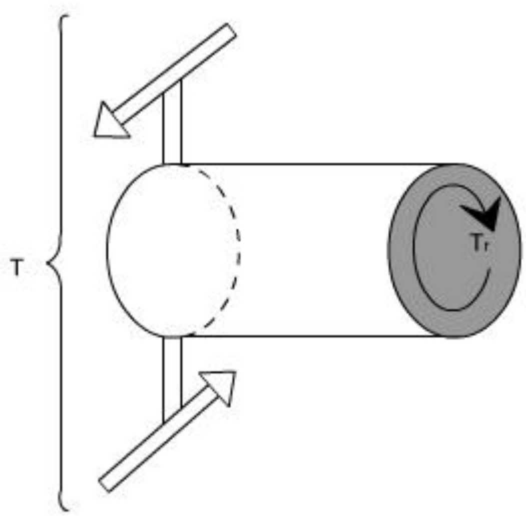


Fig 2: When the plane �mn' cuts remove the portion on R.H.S. and we get a fig 2. Now since the entire member is in equilibrium, therefore, each portion must be in equilibrium. Thus, the member is in equilibrium under the action of resultant external torque T and developed resisting Torque T_r .

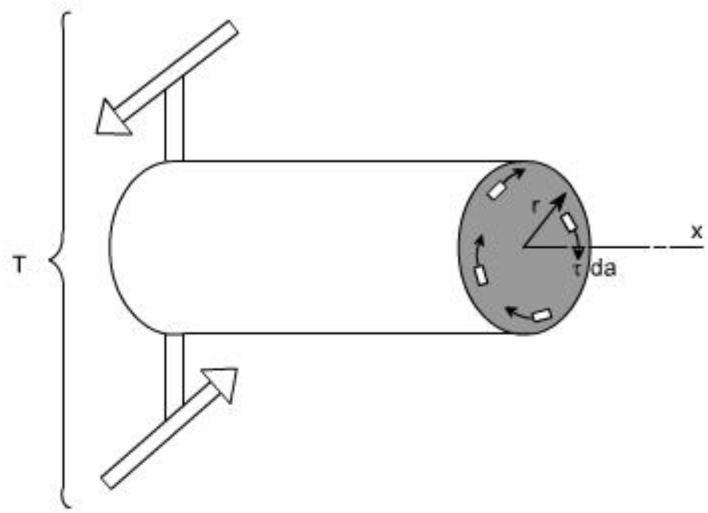


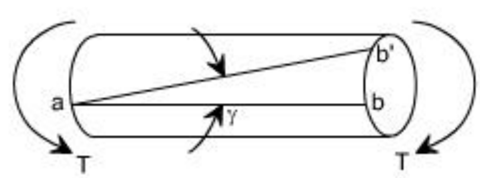
Fig 3: The Figure shows that how the resisting torque T_r is developed. The resisting torque T_r is produced by virtue of an infinites mal shear forces acting on the plane perpendicular to the axis of the shaft. Obviously such shear forces would be developed by virtue of sheer stresses.

Therefore we can say that when a particular member (say shaft in this case) is subjected to a torque, the result would be that on any element there will be shear stresses acting. While on other faces the complementary sheer forces come into picture. Thus, we can say that when a member is subjected to torque, an element of this member will be subjected to a state of pure shear.

Shaft: The shafts are the machine elements which are used to transmit power in machines.

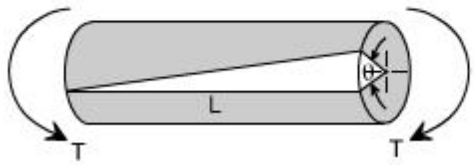
Twisting Moment: The twisting moment for any section along the bar / shaft is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section under consideration. The choice of the side in any case is of course arbitrary.

Shearing Strain: If a generator a � b is marked on the surface of the unloaded bar, then after the twisting moment 'T' has been applied this line moves to ab'. The angle � ' measured in radians, between the final and original positions of the generators is defined as the shearing strain at the surface of the bar or shaft. The same definition will hold at any interior point of the bar.



Modulus of Elasticity in shear: The ratio of the shear stress to the shear strain is called the modulus of elasticity in shear OR Modulus of Rigidity and in represented by the symbol $G = \frac{\tau}{r}$

Angle of Twist: If a shaft of length L is subjected to a constant twisting moment T along its length, than the angle □ through which one end of the bar will twist relative to the other is known is the angle of twist.

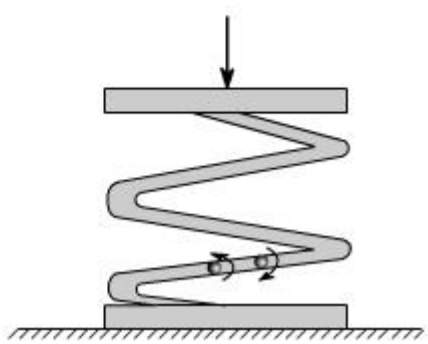


- Despite the differences in the forms of loading, we see that there are number of similarities between bending and torsion, including for example, a linear variation of stresses and strain with position.

  In torsion the members are subjected to moments (couples) in planes normal to their axes.

- For the purpose of desiging a circular shaft to withstand a given torque, we must develop an equation giving the relation between twisting moment, maximum shear stress produced, and a quantity representing the size and shape of the cross-sectional area of the shaft.

Not all torsion problems, involve rotating machinery, however, for example some types of vehicle suspension system employ torsional springs. Indeed, even coil springs are really curved members in torsion as shown in figure.



- Many torque carrying engineering members are cylindrical in shape. Examples are drive shafts, bolts and screw drivers.

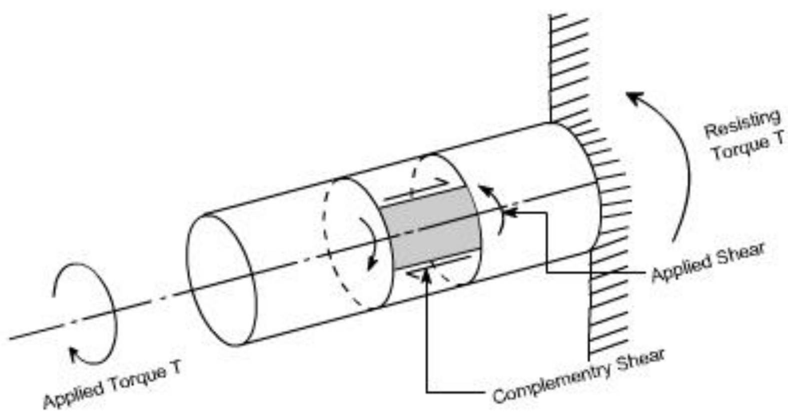
Simple Torsion Theory or Development of Torsion Formula : Here we are basically interested to derive an equation between the relevant parameters

Relationship in Torsion: $\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$

1 st Term: It refers to applied loading ad a property of section, which in the instance is the polar second moment of area.

2 nd Term: This refers to stress, and the stress increases as the distance from the axis increases.

3 rd Term: it refers to the deformation and contains the terms modulus of rigidity & combined term (□̃ l) which is equivalent to strain for the purpose of designing a circular shaft to with stand a given torque we must develop an equation giving the relation between Twisting moments max m shear stain produced and a quantity representing the size and shape of the cross � sectional area of the shaft.



Refer to the figure shown above where a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear, the moment of resistance developed by the shear stresses being every where equal to the magnitude, and opposite in sense, to the applied torque. For the purpose of deriving a simple theory to describe the behavior of shafts subjected to torque it is necessary make the following base assumptions.

Assumption:

(i) The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

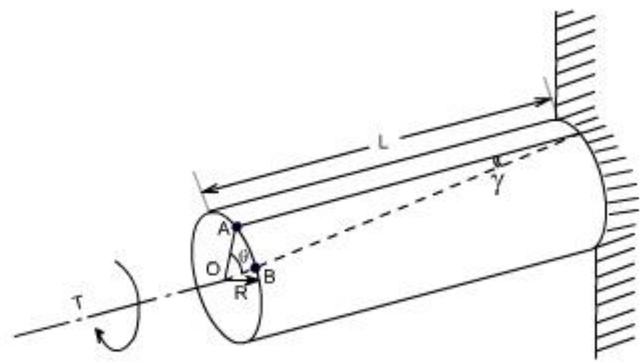
(ii) The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

(iii) The stress does not exceed the elastic limit.

(iv) The circular section remains circular

(v) Cross section remain plane.

(vi) Cross section rotate as if rigid i.e. every diameter rotates through the same angle.



Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle □ , point A moves to B, and AB subtends an angle � □ ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius

arc AB = R□

= L □ [since L and □ also constitute the arc AB]

Thus, □ = R□ / L (1)

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

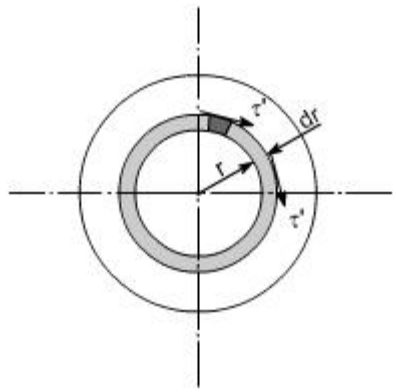
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress τ'.



The force set up on each element

= stress x area

$= \tau' \times 2\pi r\, dr$ (approximately)

This force will produce a moment or torque about the center axis of the shaft.

$= \tau' . 2\pi r\, dr . r$

$= 2\pi . \tau' . r^2 . dr$

The total torque T on the section, will be the sum of all the contributions. $T = \int_0^R 2\pi\tau' r^2 dr$

Since τ' is a function of r, because it varies with radius so writing down τ' in terms of r from the equation (1).

$$\text{i.e } \tau' = \frac{G\theta . r}{L}$$

$$\text{we get } T = \int_0^R 2\pi \frac{G\theta}{L} . r^3 \, dr$$

$$T = \frac{2\pi G\theta}{L} \int_0^R r^3 \, dr$$

$$= \frac{2\pi G\theta}{L} . \left[\frac{R^4}{4} \right]_0^R$$

$$= \frac{G\theta}{L} . \frac{2\pi R^4}{4}$$

$$= \frac{G\theta}{L} . \frac{\pi R^4}{2}$$

$$= \frac{G\theta}{L} . \left[\frac{\pi d^4}{32} \right] \text{ now substituting } R = d/2$$

$$= \frac{G\theta}{L} . J$$

$$\text{since } \frac{\pi d^4}{32} = J \text{ the polar moment of inertia}$$

$$\text{or } \frac{T}{J} = \frac{G\theta}{L} \qquad \text{......(2)}$$

if we combine the equation no.(1) and (2) we get $\boxed{\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}}$

Where

T = applied external Torque, which is constant over Length L;

J = Polar moment of Inertia

$= \frac{\pi d^4}{32}$ for solid shaft

$= \frac{\pi (D^4 - d^4)}{32}$ for a hollow shaft. [D = Outside diameter ; d = inside diameter]

G = Modules of rigidity (or Modulus of elasticity in shear)

□ = It is the angle of twist in radians on a length L.

Tensional Stiffness: The tensional stiffness k is defined as the torque per radius twist

i.e, k = T /□ □ □ = GJ / L

Power Transmitted by a shaft : If T is the applied Torque and □ is the angular velocity of the shaft, then the power transmitted by the shaft is

$$P = T.\omega = \frac{2\pi NT}{60} = \frac{2\pi NT}{60.10^3}\,\text{kw}$$

$$\text{where } N = \text{rpm}$$

Distribution of shear stresses in circular Shafts subjected to torsion :

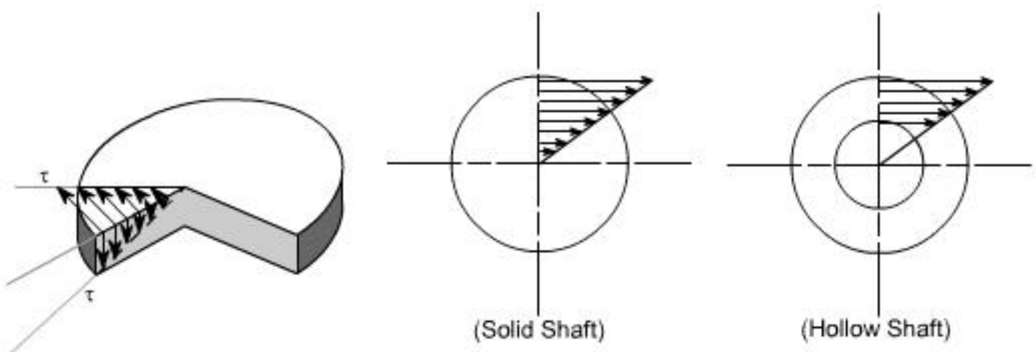
The simple torsion equation is written as

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

$$\text{or}$$

$$\tau = \frac{G\theta.r}{L}$$

This states that the shearing stress varies directly as the distance �r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum strear stress occurs on the outer surface of the shaft where r = R

The value of maximum shearing stress in the solid circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} . d/2$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

From the above relation, following conclusion can be drawn

(i) □ $_{max}{}^{m}$ □ T

(ii) □ $_{max}{}^{m}$ □ 1/d ^{3}

Power Transmitted by a shaft:

In practical application, the diameter of the shaft must sometimes be calculated from the power which it is required to transmit.

Given the power required to be transmitted, speed in rpm �N' Torque T, the formula connecting

These quantities can be derived as follows

$$P = T.\omega$$

$$= \frac{T.2\pi N}{60} \text{watts}$$

$$= \frac{2\pi NT}{60 \times 10^3} (\text{kw})$$

Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist .

$$k = \frac{T}{\theta}$$

$$\text{i.e} \quad = \frac{GJ}{L}$$

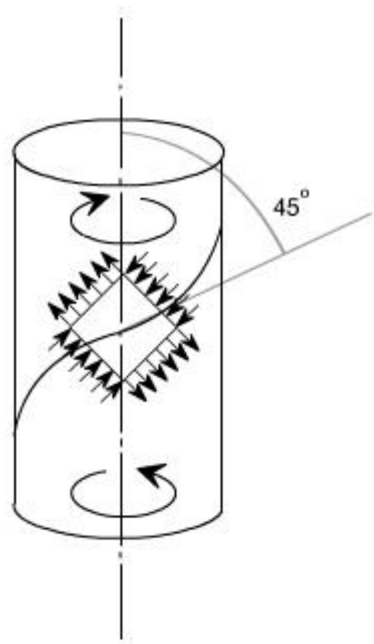
$$\text{or} \quad k = \frac{G.J}{L}$$

For a ductile material, the plastic flow begins first in the outer surface. For a material which is weaker in shear longitudinally than transversely � for instance a wooden shaft, with the fibres parallel to axis the first cracks will be produced by the shearing stresses acting in the axial section and they will upper on the surface of the shaft in the longitudinal direction.

In the case of a material which is weaker in tension than in shear. For instance a, circular shaft of cast iron or a cylindrical piece of chalk a crack along a helix inclined at 45^0 to the axis of shaft often occurs.

Explanation: This is because of the fact that the state of pure shear is equivalent to a state of stress tension in one direction and equal compression in perpendicular direction.

A rectangular element cut from the outer layer of a twisted shaft with sides at 45^0 to the axis will be subjected to such stresses, the tensile stresses shown will produce a helical crack mentioned.



<u>TORSION OF HOLLOW SHAFTS:</u>

From the torsion of solid shafts of circular x � section , it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0^4 - d_i^4)}{32} \quad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0^4 - d_i^4)}$$

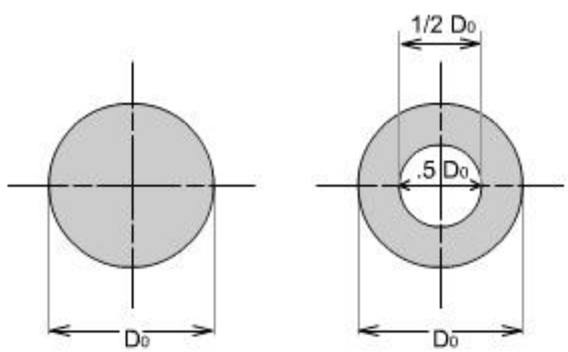
$$= \frac{16T.D_0}{\pi D_0^4\left[1-(d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0^3\left[1-(1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0^3} \qquad (2)$$

Hence by examining the equation (1) and (2) it may be seen that the □ $_{max}^{m}$ in the case of hollow shaft is 6.6% larger then in the case of a solid shaft having the same outside diameter.

Reduction in weight:

Considering a solid and hollow shafts of the same length 'l' and density '□' with d_i = 1/2 D_o



Weight of hollow shaft

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi (D_0/2)^2}{4}\right] l \times \rho$$

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi D_0^2}{16}\right] l \times \rho$$

$$= \frac{\pi D_0^2}{4}\left[1 - 1/4\right] l \times \rho$$

$$= 0.75 \frac{\pi D_0^2}{4} l \times \rho$$

$$\text{Weight of solid shaft} = \frac{\pi D_0^2}{4} . l . \rho$$

$$\text{Reduction in weight} = (1 - 0.75) \frac{\pi D_0^2}{4} l \times \rho$$

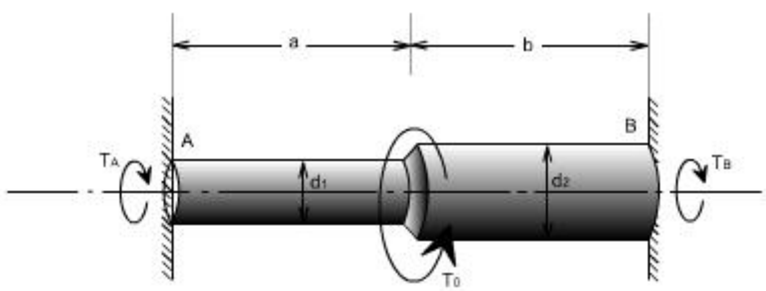
$$= 0.25 \frac{\pi D_0^2}{4} l \times \rho$$

Hence the reduction in weight would be just 25%.

<u>Illustrative Examples :</u>

<u>Problem 1</u>

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T_0 at the shoulder as shown in the figure. Determine the angle of rotation □$_0$ of the shoulder section where T_0 is applied ?



Solution: This is a statically indeterminate system because the shaft is built in at both ends. All that we can find from the statics is that the sum of two reactive torque T_A and T_B at the built � in ends of the shafts must be equal to the applied torque T_0

Thus $T_A + T_B = T_0$ ------ (1)

[from static principles]

Where T_A ,T_B are the reactive torque at the built in ends A and B. wheeras T_0 is the applied torque

From consideration of consistent deformation, we see that the angle of twist in each portion of the shaft must be same.

i.e $\theta_a = \theta_b = \theta_0$

using the relation for angle of twist

$$\frac{T}{J} = \frac{G.\theta}{l}$$

$$\text{or } \theta_A = \frac{T_A a}{J_A G}$$

$$\theta_B = \frac{T_B a}{J_B G}$$

$$\Rightarrow \frac{T_A a}{J_A G} = \frac{T_B b}{J_B G} = \theta_0 \quad \text{or} \quad \frac{T_A}{T_B} = \frac{J_A}{J_B}.\frac{b}{a} \quad (2)$$

N.B: Assuming modulus of rigidity G to be same for the two portions

So the defines the ratio of T_A and T_B

So by solving (1) & (2) we get

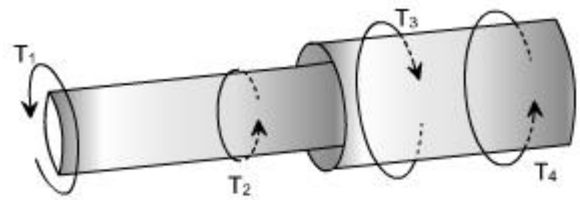
$$T_A = \frac{T_0}{1 + \frac{J_B a}{J_A b}}$$

$$T_b = \frac{T_0}{1 + \frac{J_a b}{J_b a}}$$

Using either of these values in (2) we have the angle of rotation θ_0 at the junction

$$\theta_0 = \frac{T_0.a.b}{[J_A.b + J_B.a]G}$$

Non Uniform Torsion: The pure torsion refers to a torsion of a prismatic bar subjected to torques acting only at the ends. While the non uniform torsion differs from pure torsion in a sense that the bar / shaft need not to be prismatic and the applied torques may vary along the length.



Here the shaft is made up of two different segments of different diameters and having torques applied at several cross sections. Each region of the bar between the applied loads between changes in cross section is in pure torsion, hence the formula's derived earlier may be applied. Then form the internal torque, maximum shear stress and angle of rotation for each region can be calculated from the relation

$$\frac{T}{J} = \frac{\tau}{r} \text{ and } \frac{T}{J} = \frac{G\theta}{L}$$

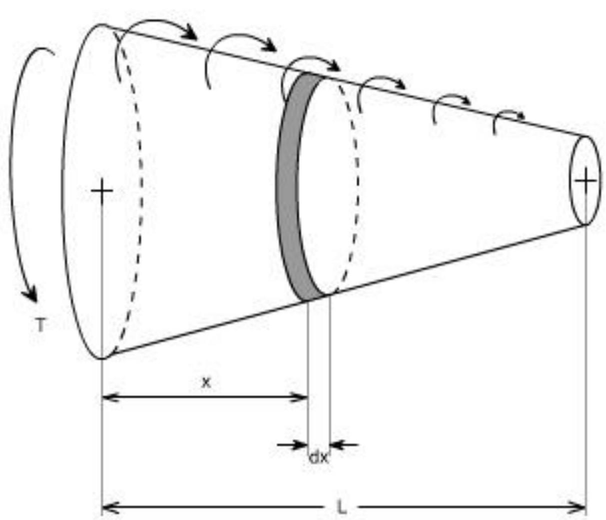
The total angle to twist of one end of the bar with respect to the other is obtained by summation using the formula

$$\theta = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$

i = index for no. of parts

n = total number of parts

If either the torque or the cross section changes continuously along the axis of the bar, then the (summation can be replaced by an integral sign (∫). i.e We will have to consider a differential element.



After considering the differential element, we can write $d\theta = \frac{T_x dx}{GI_x}$

Substituting the expressions for T_x and J_x at a distance x from the end of the bar, and then integrating between the limits 0 to L, find the value of angle of twist may be determined.

$$\theta = \int_0^L d\theta = \int_0^L \frac{T_x dx}{GI_x}$$

Closed Coiled helical springs subjected to axial loads:

Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released.

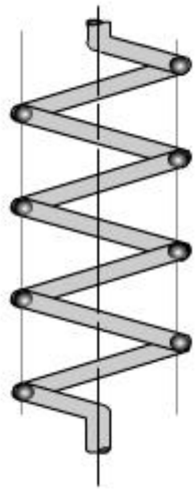
or

Springs are energy absorbing units whose function is to store energy and to restore it slowly or rapidly depending on the particular application.

Important types of springs are:

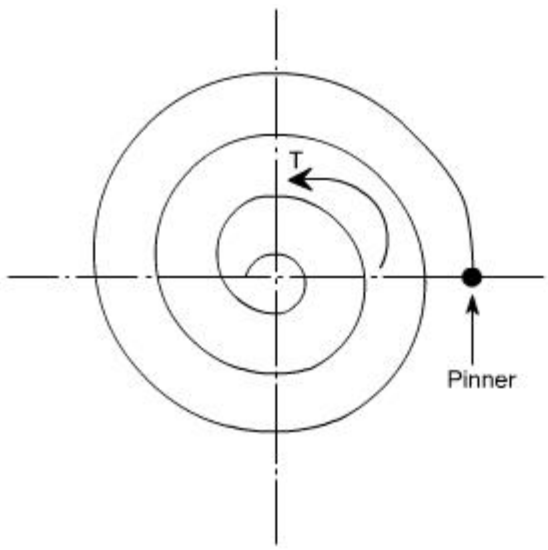
There are various types of springs such as

(i) helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is torsional shear stress due to twisting. They are both used in tension and compression.

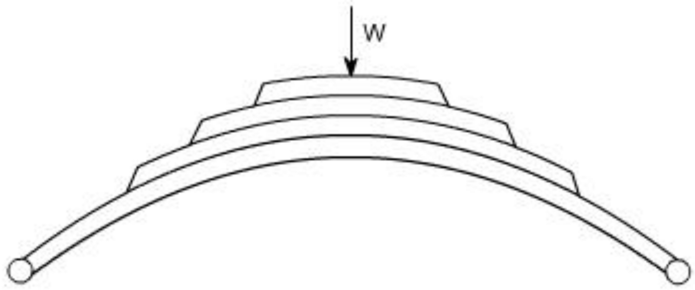


(ii) Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.

In this the major stresses are tensile and compression due to bending.



(iv) Leaf springs: They are composed of flat bars of varying lengths clamped together so as to obtain greater efficiency . Leaf springs may be full elliptic, semi elliptic or cantilever types, In these type of springs the major stresses which come into picture are tensile & compressive.



These type of springs are used in the automobile suspension system.

Uses of springs :

(a) To apply forces and to control motions as in brakes and clutches.

(b) To measure forces as in spring balance.

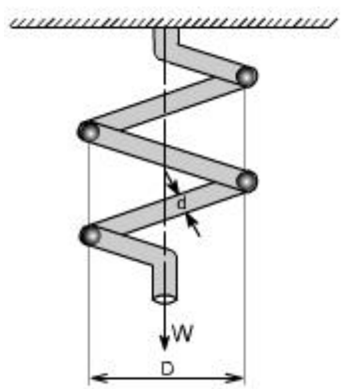
(c) To store energy as in clock springs.

(d) To reduce the effect of shock or impact loading as in carriage springs.

(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

Derivation of the Formula :

In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load W.



Let

W = axial load

D = mean coil diameter

d = diameter of spring wire

n = number of active coils

C = spring index = D / d For circular wires

l = length of spring wire

G = modulus of rigidity

x = deflection of spring

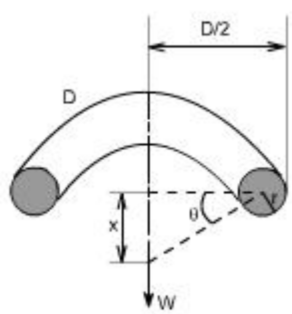
q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that

x = D / 2 . □

again l = □ D n [consider ,one half turn of a close coiled helical spring]



Assumptions: (1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a such a spring will be assumed to lie in a plane which is nearly □^{r} to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical. Hence to maintain equilibrium of a segment of the spring, only a shearing force V = F and Torque T = F. r are required at any X � section. In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is uniformly distributed and is negligible

so applying the torsion formula.

Using the torsion formula i.e

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

and substitituting $J = \frac{\pi d^4}{32}; T = w.\frac{d}{2}$

$$\theta = \frac{2.x}{D}; l = \pi D.x$$

<u>SPRING DEFLECTION</u>

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{G.2x/D}{\pi D.n}$$

Thus,

$$x = \frac{8w.D^3.n}{G.d^4}$$

Spring striffness: The stiffness is defined as the load per unit deflection therefore

$$k = \frac{w}{x} = \frac{w}{\frac{8w.D^3.n}{G.d^4}}$$

Therefore

$$k = \frac{G.d^4}{8.D^3.n}$$

<u>Shear stress</u>

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{\tau_{max^m}}{d/2}$$

or $\tau_{max^m} = \frac{8wD}{\pi d^3}$

WAHL'S FACTOR :

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

K = Wahl' s factor and is defined as $K = \frac{4c - 1}{4c - 4} + \frac{0.615}{c}$

Where C = spring index

= D/d

if we take into account the Wahl's factor than the formula for the shear stress becomes $\tau_{max^m} = \frac{16.T.k}{\pi d^3}$

Strain Energy : The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion

$$U = \frac{T^2 L}{2EI}$$

$$L = \pi D n$$

$$I = \frac{\pi d^4}{64}$$

so after substitution we get

$$U = \frac{32 T^2 D n}{E.d^4}$$

Example: A close coiled helical spring is to carry a load of 5000N with a deflection of 50 mm and a maximum shearing stress of 400 N/mm^2 .if the number of active turns or active coils is 8.Estimate the following:

(i) wire diameter

(ii) mean coil diameter

(iii) weight of the spring.

Assume G = 83,000 N/mm^2 ; □ = 7700 kg/m^3

solution :

(i) for wire diametre if W is the axial load, then

$$\frac{\frac{w.d/2}{\frac{\pi d^4}{32}}}{} = \frac{\tau_{max^m}}{d/2}$$

$$D = \frac{400}{d/2}.\frac{\pi d^4}{32}.\frac{2}{W}$$

$$D = \frac{400.\pi d^3.2}{5000.16}$$

$$D = 0.0314\, d^3$$

Futher, deflection is given as

$$x = \frac{8wD^3.n}{G.d^4}$$

on substituting the relevant parameters we get

$$50 = \frac{8.5000.(0.0314\, d^3)^3.8}{83{,}000.d^4}$$

$$d = 13.32mm$$

Therefore,

$D = .0314 \times (13.317)^3 mm$

$= 74.15mm$

D = 74.15 mm

<u>Weight</u>

mass or weight = volume. density

= area.length of the spring.density of spring material

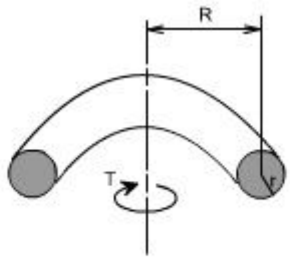
$$= \frac{\pi d^2}{4}.\pi D n.\rho$$

On substituting the relevant parameters we get

Weight = 1.996 kg

= 2.0kg

<u>Close � coiled helical spring subjected to axial torque T or axial couple.</u>



In this case the material of the spring is subjected to pure bending which tends to reduce Radius R of the coils. In this case the bending moment is constant through out the spring and is equal to the applied axial Torque T. The stresses i.e. maximum bending stress may thus be determined

$$\sigma_{max} = \frac{M.y}{I}$$

$$= \frac{T.d/2}{\frac{\pi d^4}{64}}$$

from the bending theory. $$\sigma_{max} = \frac{32T}{\pi d^3}$$

Deflection or wind � up angle:

Under the action of an axial torque the deflection of the spring becomes the �wind � up� angle of the spring which is the angle through which one end turns relative to the other. This will be equal to the total change of slope along the wire, according to area � moment theorem

$$\theta = \int_0^L \frac{MdL}{EI} \text{ but } M = T$$

$$= \int_0^L \frac{T.dL}{EI} = \frac{T}{EI}\int_0^L dL$$

Thus, as'T 'remainsconstant

$$\theta = \frac{T.L}{EI}$$

Futher

$$L = \pi D.n$$

$$I = \frac{\pi d^4}{64}$$

Therefore, on substitution,the value of θ obtained is

$$\boxed{\theta = \frac{64T.D.n}{E.d^4}}$$