LECTURE 8 Two-dimensional (Plane) Stress State. Graphical Method of Stress State Analysis

1 Two-Dimensional (Plane) Stress State Definition and Examples

A two-dimensional (plane) state of stress exists at a point of deformable solid, when the stresses are independent of one of the three coordinate axis. It means that ***the general feature of this type of stress state is the presence of one zero principal plane***. Examples include the stresses arising on inclined sections of an axially loaded rod (Fig. 1), a shaft in torsion (Fig. 2), a beam at combined loading (Fig. 3), thin-walled vessel under internal pressure p (Fig. 4), aircraft wind skin (Fig. 5), scoop box (Fig. 6).

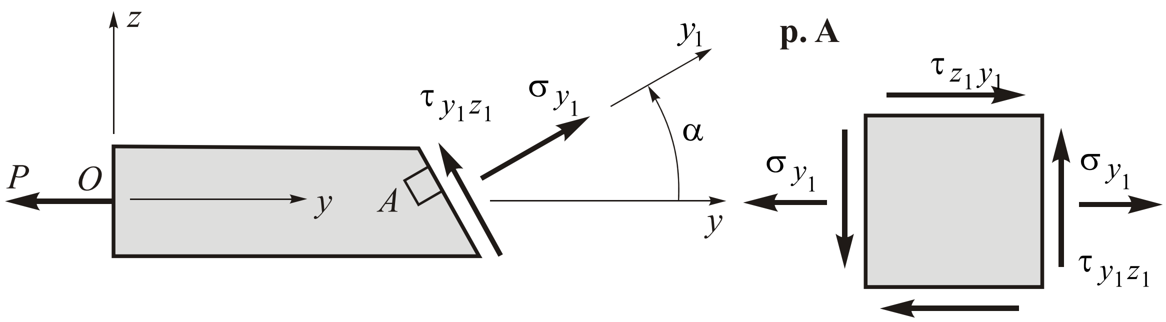


Fig. 1 Two-dimensional stresses on inclined section in axial loading

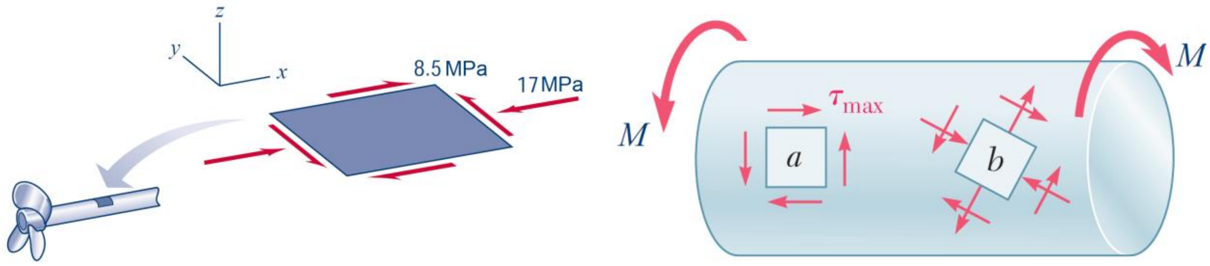


Fig. 2 Plane stress state at surface points of the shaft in torsion

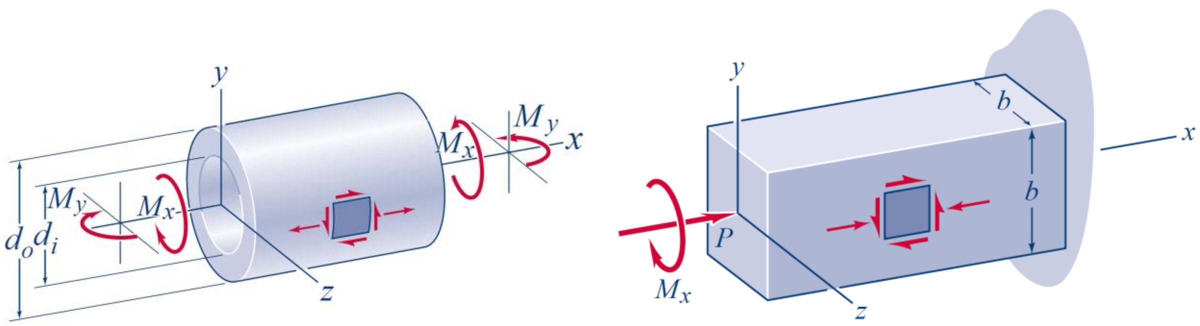


Fig. 3 Plane stress state at the surface point of a bar under combined loading

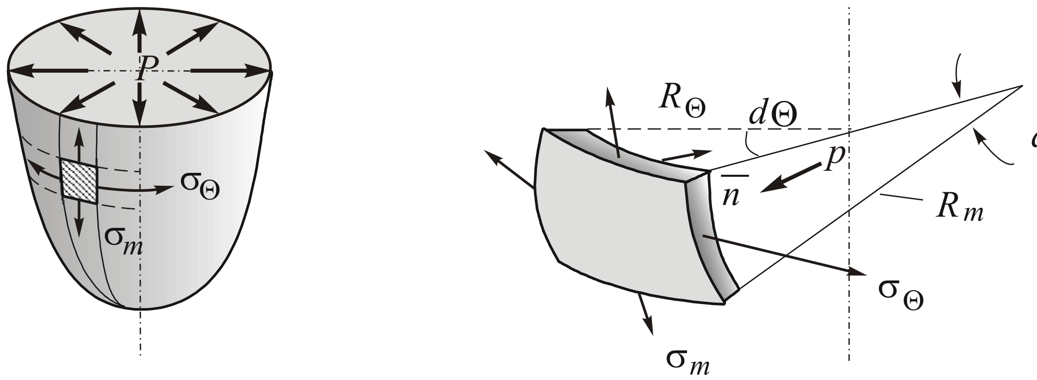


Fig. 4 Element in plane biaxial stress state in pressure vessel

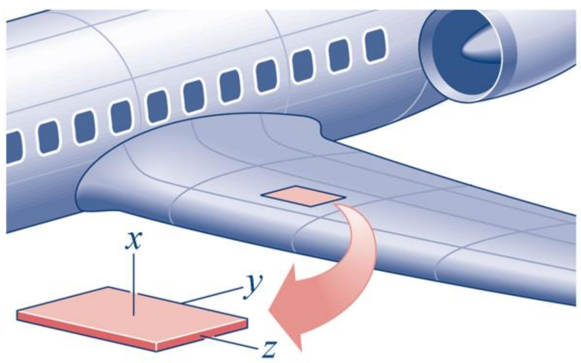


Fig. 5 Two-dimensional stress state in the point of wing skin (stresses, normal to skin surface, are zero)

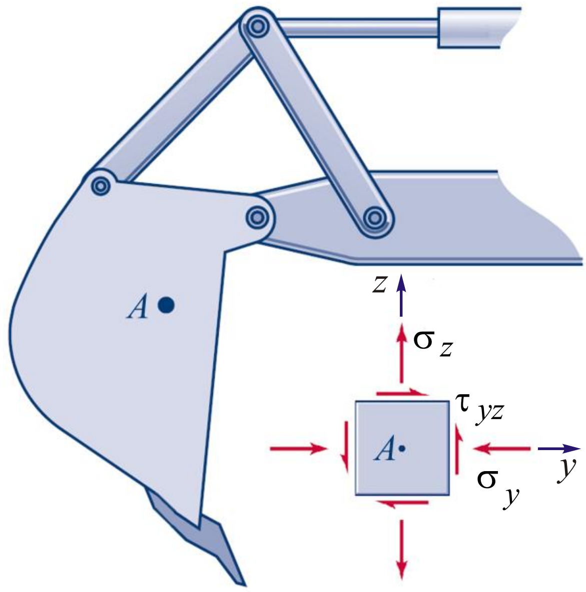


Fig. 6 Plane stress state in the point of scoop box

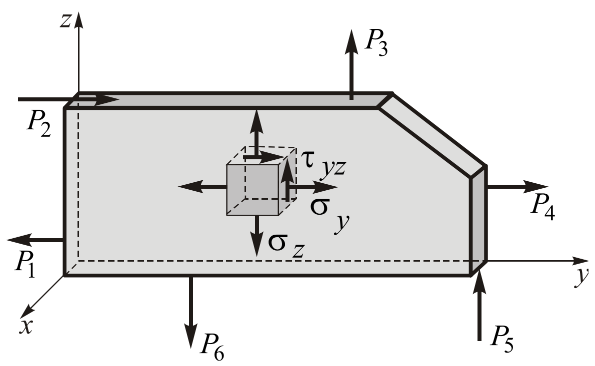


Fig.7 Thin plane subjected to plane stress

Two-dimensional problems are of two classes: **plane stress** and **plane strain**. The condition that occurs in a **thin plate** subjected to loading uniformly distributed over the thickness and parallel to the plane of plate typifies the **state of plane stress** (plane stressed state, plane stress) (Fig. 7). Because the plate is thin, the stress-distribution may be closely approximated by assuming that two-dimensional stress components do not vary throughout the thickness and the other components are zero. Another case of plane stress exists on the free surface of a structural or machine component (see Figs. 2, 3, 5, 6).

To explain plane stress, we will consider the stress element shown in Fig. 8. This element is infinitesimal in size and can be sketched either as a cube or as a rectangular parallelepiped.

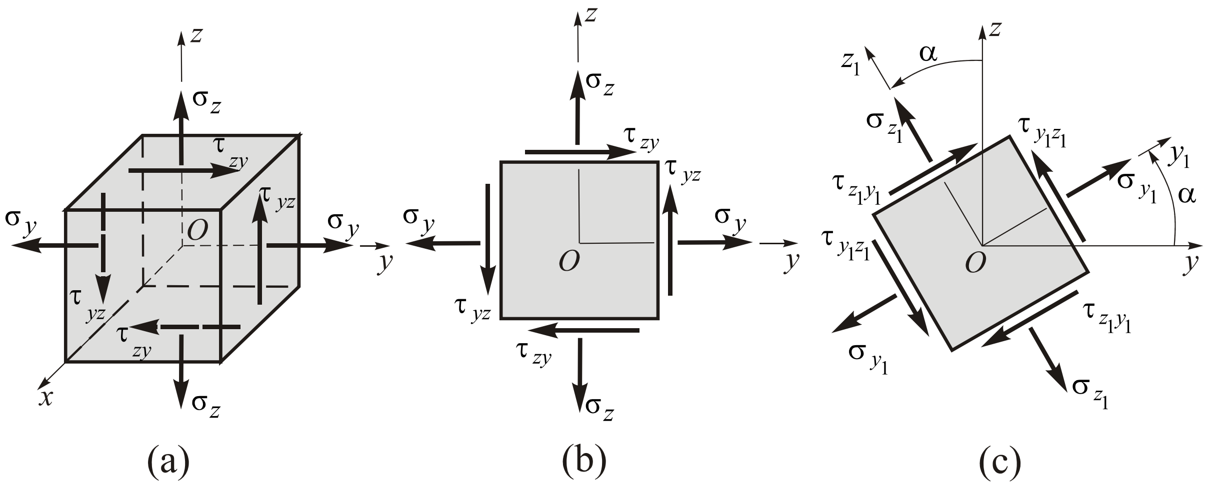


Fig. 8 Elements in plane stress: (a) three-dimensional view of an element oriented to the x, y, z axes, (b) two-dimensional view of the same element, and (c) two-dimensional view of an element oriented to the x_1, y_1, z_1 axes

The *xyz* axes are parallel to the edges of the element, and the faces of the element are designated by the directions of their outward normals. For instance, the **right-hand face** of the element is referred to as the **positive *y* face**, and the left-hand face (hidden from the viewer) is referred to as the negative *y* face. Similarly, the **top face is the positive *z* face**, and the **front face is the positive *x* face**.

When the material is in plane stress in the *yz* plane, only the *y* and *z* faces of the element are subjected to stresses, and all stresses act parallel to the *y* and *z* axes, as shown in Fig. 8a. This stress condition is very common because ***it exists at the surface of any stressed body***, except at points where external loads act on the surface. When the element shown in Fig. 8a is located at the free surface of a body, the *x* face is in the plane of the surface (no stresses) and the *x* axis is normal to the surface. This face may be considered as **zero principal plane** (see Fig. 5).

The symbols for the stresses shown in Fig. 8a have the following meanings. A normal stress σ has a subscript that identifies the face on which the stress acts; for instance, the stress σ_y acts on the *y* face of the element and the stress σ_z acts on the *z* face of the element. ***Since the element is infinitesimal in size, equal normal stresses act on the opposite faces***. The sign convention for normal stresses is the familiar one, namely, tension is positive and compression is negative.

A shear stress τ has two subscripts – the first subscript denotes the normal to the face on which the stress acts, and the second gives the direction on that face. Thus, the stress τ_{yz} acts on the *y* face in the direction of the *z* axis (Fig. 8a), and stress τ_{zy} acts on the *z* face in the direction of the *y* axis.

The sign convention for shear stresses is as follows. A ***shear stress is positive when it acts on a positive face of an element in the positive direction of an axis, and it is negative when it acts on a positive face of an element in the negative direction of an axis***. Therefore, the stresses τ_{yz} and τ_{zy} shown on the positive *y* and *z* faces in Fig. 8a are positive shear stresses. Similarly, on a negative face of the element, a shear stress is positive when it acts in the negative direction of an axis. Hence, the stresses τ_{yz} and τ_{zy} shown on the negative *y* and *z* faces of the element are also positive.

The preceding sign convention for shear stresses is dependable on the equilibrium of the element, because we know that shear stresses on opposite faces of an infinitesimal element must be equal in magnitude and opposite in direction. Hence, according to our sign convention, a positive stress τ_{yz} acts upward on the positive face (Fig. 8a) and downward on the negative face. In a similar manner, the stresses τ_{zy} acting on the top and bottom faces of the element are positive although they have opposite directions.

We know that shear stresses on mutually perpendicular planes are equal in magnitude and have directions such that ***both stresses point toward***, or ***both point away from, the line of intersection of the faces***. Inasmuch as τ_{yz} and τ_{zy} are positive in the directions shown in the Fig. 8, they are consistent with this observation. Therefore, we note that

$$\tau_{yz} = \tau_{zy}. \tag{1}$$

This equation was called earlier the **law of equality for shear stresses**. It was derived from equilibrium of the element.

For convenience in sketching plane-stress elements, we usually draw only a two-dimensional view of the element, as shown in Fig. 8b.

2 Stresses on Inclined Planes

Our goal now is to consider the stresses acting on inclined sections, assuming that the stresses σ_y, σ_z, and τ_{yz} (Figs. 8a and b) are known. To determine the stresses acting on an inclined section at positive (counterclockwised) α-angle, we consider a new stress element (Fig. 8c) that is located at the same point in the material as the original element (Fig. 8b). However, the new element has faces that are parallel and perpendicular to the inclined direction. Associated with this new element are axes y_1, z_1 and x_1 such that the x_1 axis coincides with the *x* axis and the y_1, z_1 ***axes are rotated counterclockwise through an angle α with respect to the yz axes***. The normal and shear stresses acting on this new element are denoted σ_{y_1}, σ_{z_1}, $\tau_{y_1 z_1}$, and $\tau_{z_1 y_1}$, using the same subscript designations and sign conventions described above for the stresses acting on the *yz* element. The previous conclusions regarding the shear stresses still apply, so that

$$\tau_{y_1 z_1} = \tau_{z_1 y_1}. \tag{2}$$

Note, that more simple designation of the stresses on inclined faces is used: $\sigma_{y_1} = \sigma_\alpha$, $\sigma_{z_1} = \sigma_\beta$, $\tau_{yz} = \tau_\alpha$, $\tau_{zy} = \tau_\beta$.

From Eq. 2 and the equilibrium of the element, we see that the ***shear stresses acting on all four side faces of an element in plane stress are known if we determine the shear stress acting on any one of those faces***.

The stresses acting on the inclined y_1, z_1 element (Fig. 8c) can be expressed in terms of the stresses on the *yz* element (Fig. 8b) by using **equations of equilibrium**. For this purpose, we choose a **wedge-shaped stress element** (Fig. 9a) having an inclined face that is the same as the y_1 face of the inclined element. The other two side faces of the wedge are parallel to the *y* and *z* axes.

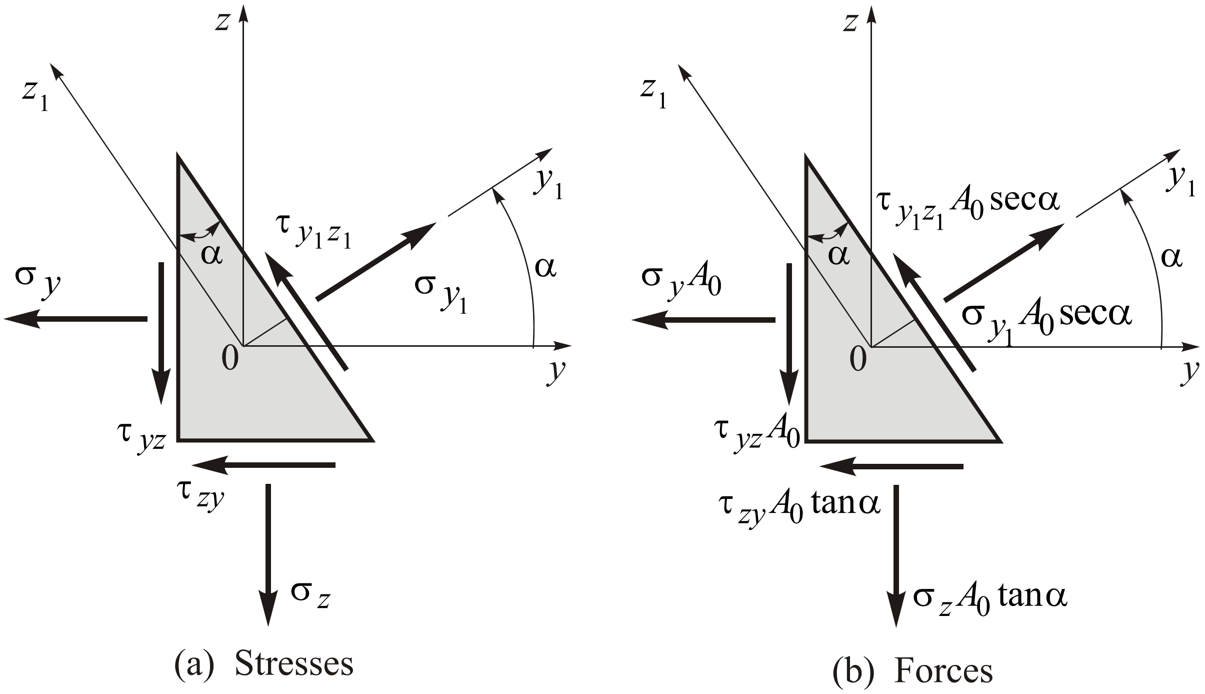


Fig. 9 Wedge-shaped stress element in plane stress state: (a) stresses acting on the element, and (b) internal forces acting on the element

In order to write equations of equilibrium for the wedge, we need to construct a free-body diagram showing the forces acting on the faces. Let us denote the area of the left-hand side face (that is, the negative *y* face) as A_0. Then the normal and shear forces acting on that face are $\sigma_y A_0$ and $\tau_{yz} A_0$, as shown in the free-body diagram of Fig. 9b. The area of the bottom face (or negative *z* face) is $A_0 \tan\alpha$, and the area of the inclined face (or positive y_1 face) is $A_0 \sec\alpha$. Thus, the normal and shear forces acting on these faces have the magnitudes and directions shown in the Fig. 9b.

The forces acting on the left-hand and bottom faces can be resolved into orthogonal components acting in the y_1 and z_1 directions. Then we can obtain two equations of equilibrium by summing forces in those directions. The first equation, obtained by summing forces in the y_1 direction, is

$$\begin{aligned}&\sigma_{y_1} A_0 \sec\alpha - \sigma_y A_0 \cos\alpha - \tau_{yz} A_0 \sin\alpha - \\ &-\sigma_z A_0 \tan\alpha \sin\alpha - \tau_{zy} A_0 \tan\alpha \cos\alpha = 0.\end{aligned} \tag{3}$$

Summation of forces in the y_1 direction gives

$$\begin{aligned}&\tau_{y_1 z_1} A_0 \sec\alpha + \sigma_y A_0 \sin\alpha - \tau_{yz} A_0 \cos\alpha - \\ &-\sigma_z A_0 \tan\alpha \cos\alpha + \tau_{zy} A_0 \tan\alpha \sin\alpha = 0.\end{aligned} \tag{4}$$

Using the relationship $\tau_{yz} = \tau_{zy}$, we obtain after simplification the following two equations:

$$\sigma_{y_1} = \sigma_y \cos^2\alpha + \sigma_z \sin^2\alpha + 2\tau_{yz} \sin\alpha \cos\alpha\,, \tag{5}$$

$$\tau_{y_1 z_1} = -\left(\sigma_y - \sigma_z\right)\sin\alpha\cos\alpha + \tau_{yz}\left(\cos^2\alpha - \sin^2\alpha\right). \tag{6}$$

Equations (5) and (6) give the normal and shear stresses acting on the y_1 plane in terms of the angle α and the stresses σ_y, σ_z, and τ_{yz} acting on the *y* and *z* planes. Due to σ_{y_1} and $\tau_{y_1 z_1}$ are applied to the inclined face at the α angle relative to *y* direction, it is convenient to designate, that

$$\sigma_{y_1} = \sigma_\alpha \quad \text{and} \quad \tau_{y_1 z_1} = \tau_\alpha\,. \tag{7}$$

It is interesting to note, that in $\alpha = 0$ Eqs. (5) and (6) give $\sigma_{y_1} = \sigma_y$ and $\tau_{y_1 z_1} = \tau_{yz}$. Also, when $\alpha = 90°$, these equations give $\sigma_{y_1} = \sigma_z$ and $\tau_{y_1 z_1} = -\tau_{yz} = -\tau_{zy}$. In the latter case, since the y_1 axis is vertical when $\alpha = 90°$, the stress $\tau_{y_1 z_1}$ will be positive when it acts to the left. However, the stress τ_{zy} acts to the right, and therefore $\tau_{y_1 z_1} = -\tau_{zy}$.

Equations (5) and (6) can be expressed in a more convenient form by introducing the following trigonometric identities:

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha), \quad \sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha), \quad \sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha . \quad (8)$$

After these substitutions the equations become

$$\sigma_{y_1} = \frac{\sigma_y + \sigma_z}{2} + \frac{\sigma_y - \sigma_z}{2}\cos 2\alpha + \tau_{yz}\sin 2\alpha , \quad (9)$$

$$\tau_{y_1 z_1} = -\frac{\sigma_y - \sigma_z}{2}\sin 2\alpha + \tau_{yz}\cos 2\alpha . \quad (10)$$

These equations are known as the **transformation equations for plane stress** because they transform the stress components from one set of axes to another.

Note. (1) Only one intrinsic state of stress exists at the point in a stressed body, regardless of the orientation of the element, i.e. whether represented by stresses acting on the yz element (Fig. 8b) or by stresses acting on the inclined $y_1 z_1$ element (Fig. 8c). (2) Since the transformation equations were derived only from equilibrium of an element, they are applicable to stresses in any kind of material, whether linear or nonlinear, elastic or inelastic.

An important result concerning the normal stresses can be obtained from the transformation equations. The normal stress σ_{z_1} acting on the z_1 face of the inclined element (Fig. 8c) can be obtained from Eq. (9) by substituting $\alpha + 90°$ for α. The result is the following equation for σ_{z_1}:

$$\sigma_{z_1} = \frac{\sigma_y + \sigma_z}{2} - \frac{\sigma_y - \sigma_z}{2}\cos 2\alpha - \tau_{yz}\sin 2\alpha . \quad (11)$$

Summing the expressions for σ_{y_1} and σ_{z_1} (Eqs. (9) and (11)), we obtain the following equation for plane stress:

$$\sigma_{y_1} + \sigma_{z_1} = \sigma_y + \sigma_z = \text{const} . \quad (12)$$

Note. The sum of the normal stresses acting on perpendicular faces of plane-stress elements (at a given point in a stressed body) is constant and independent on the angle α.

The graphs of the normal and shear stresses varying are shown in Fig. 10, which are the graphs of σ_{y_1} and $\tau_{y_1 z_1}$ versus the angle α (from Eqs. 9 and 10). The graphs are plotted for the particular case of $\sigma_z = 0.2\sigma_y$ and $\tau_{yz} = 0.8\sigma_y$. It is seen from the plots that the stresses vary continuously as the orientation of the element is changed. At certain angles, the normal stress reaches a maximum or minimum value; at other angles, it becomes zero. Similarly, the shear stress has maximum, minimum, and zero values at certain angles.

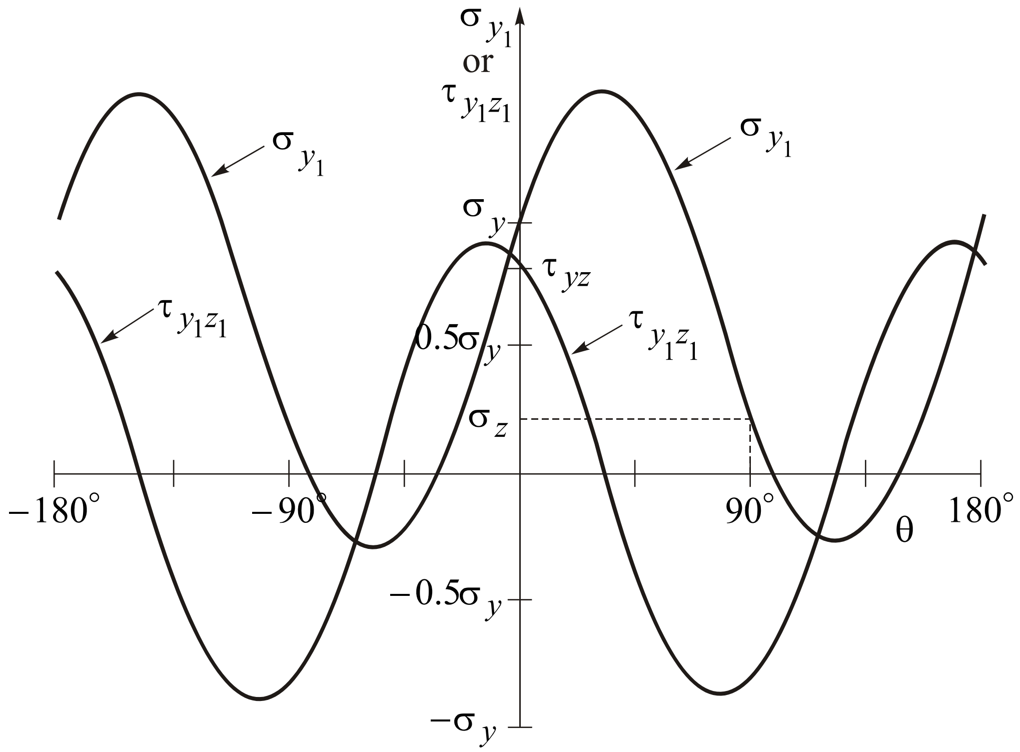


Fig. 10 Graphs of normal stress σ_{y_1} and shear stress $\tau_{y_1 z_1}$ versus the angle α (for particular case: $\sigma_z = 0.2\sigma_y$ and $\tau_{yz} = 0.8\sigma_y$)

3 Special Cases of Plane Stress

3.1 Uniaxial Stress State as a Simplified Case of Plane Stress

The general case of plane stress reduces to simpler states of stress under special conditions. For instance, as previously discussed, if all stresses acting on the *yz* element (Fig. 8b) are zero except for the normal stress σ_y, then the element is in **uniaxial stress** (Fig. 11). The corresponding transformation

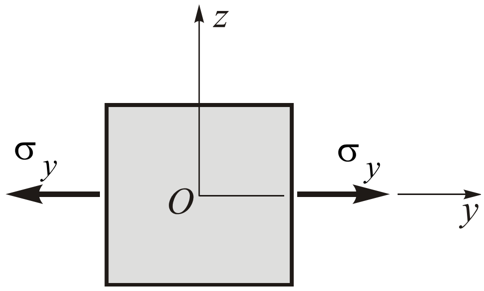


Fig. 11 Element in uniaxial stress

equations, obtained by setting σ_z and τ_{yz} equal to zero in Eqs. (9) and (10), are

$$\sigma_{y_1} = \frac{\sigma_y}{2}\left(1+\cos 2\alpha\right) = \sigma_y \cos^2\alpha\,, \tag{13}$$

$$\tau_{y_1 z_1} = -\frac{\sigma_y}{2}\left(\sin 2\alpha\right). \tag{14}$$

Note, that this type of stress state corresponds to axial tension deformation (see Fig. 1)

3.2 Pure Shear as a Special Case of Plane Stress

Pure shear is another special case of plane stress state (Fig. 12), for which the transformation equations are obtained by substituting $\sigma_y = 0$ and $\sigma_z = 0$ into Eqs. (9) and (10):

$$\sigma_{y_1} = \tau_{yz} \sin 2\alpha\,, \tag{15}$$

$$\tau_{y_1 z_1} = \tau_{yz} \cos 2\alpha\,. \tag{16}$$

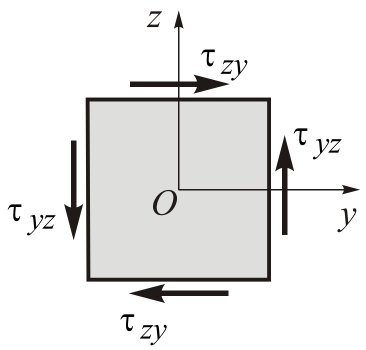


Fig. 12 Element in pure shear

3.3 Biaxial Stress

The next special case of plane stress state is called **biaxial stress**, in which the *yz* element is subjected to normal stresses in both the *y* and *z* directions but ***without any shear stresses*** (Fig. 13). The equations for biaxial stress are obtained from Eqs. (9) and (10) by dropping the terms containing τ_{yz}:

$$\sigma_{y_1} = \frac{\sigma_y + \sigma_z}{2} + \frac{\sigma_y - \sigma_z}{2}\cos 2\alpha\,, \tag{17}$$

$$\tau_{y_1 z_1} = -\frac{\sigma_y - \sigma_z}{2}\sin 2\alpha\,. \tag{18}$$

or in α, β designation,

$$\sigma_{y_1} = \sigma_y \cos^2\alpha + \sigma_z \sin^2\alpha\,, \tag{19}$$

$$\tau_{y_1 z_1} = -\frac{\left(\sigma_y - \sigma_z\right)}{2} \sin 2\alpha \, . \tag{20}$$

Biaxial stress occurs in many kinds of structures, including **thin-walled pressure vessels** (see Fig. 14).

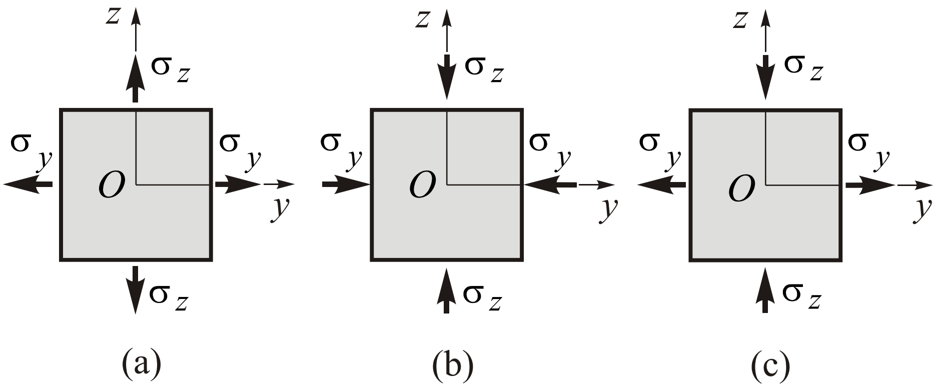


Fig. 13 Elements in biaxial stress

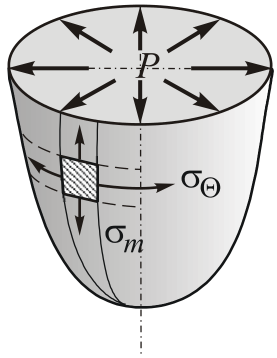
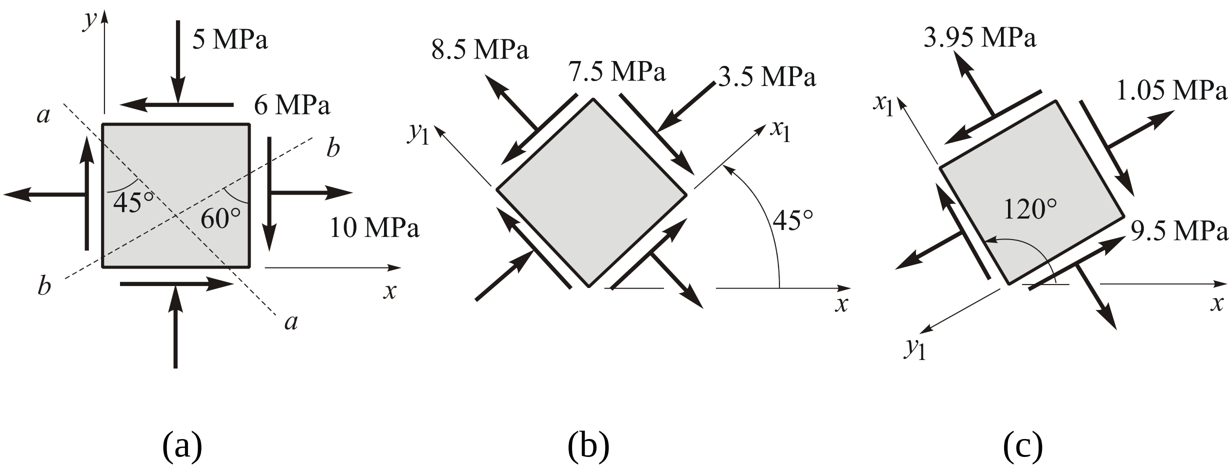


Fig. 14 Element in biaxial stress state in pressure vessel (stresses, normal to the surface are assumed to be zero)

Example 1

The state of stress at a point in the machine element is shown in Fig. a. Determine the normal and shearing stresses acting on an inclined plane parallel to (1) line $a-a$ and (2) line $b-b$.



Solution The x_1 direction is that of a normal to the inclined plane. We want to obtain the transformation of stress from the xy system of coordinates to the $x_1 y_1$ system.

Note, that the stresses and the rotations must be designated with their correct signs.

(1) Applying Eqs. (9 through 11) for $\alpha = 45°$, $\sigma_x = 10$ MPa, $\sigma_y = -5$ MPa, and $\tau_{xy} = -6$ MPa, we obtain

$$\sigma_{x_1} = \frac{1}{2}(10-5) + \frac{1}{2}(10+5)\cos 90° - 6\sin 90° = -3.5\,\text{MPa},$$

$$\tau_{x_1 y_1} = -\frac{1}{2}(10+5)\sin 90° - 6\cos 90° = -7.5\,\text{MPa},$$

and

$$\sigma_{y_1} = \frac{1}{2}(10-5) - \frac{1}{2}(10+5)\cos 90° + 6\sin 90° = 8.5\,\text{MPa}.$$

The results are indicated in Fig. b.

(2) As $\alpha = 30 + 90 = 120°$, from Eqs. (9 through 11), we have

$$\sigma_{x_1} = \frac{1}{2}(10-5) + \frac{1}{2}(10+5)\cos 240° - 6\sin 240° = 3.95\,\text{MPa},$$

$$\tau_{x_1 y_1} = -\frac{1}{2}(10+5)\sin 240° - 6\cos 240° = 9.5\,\text{MPa},$$

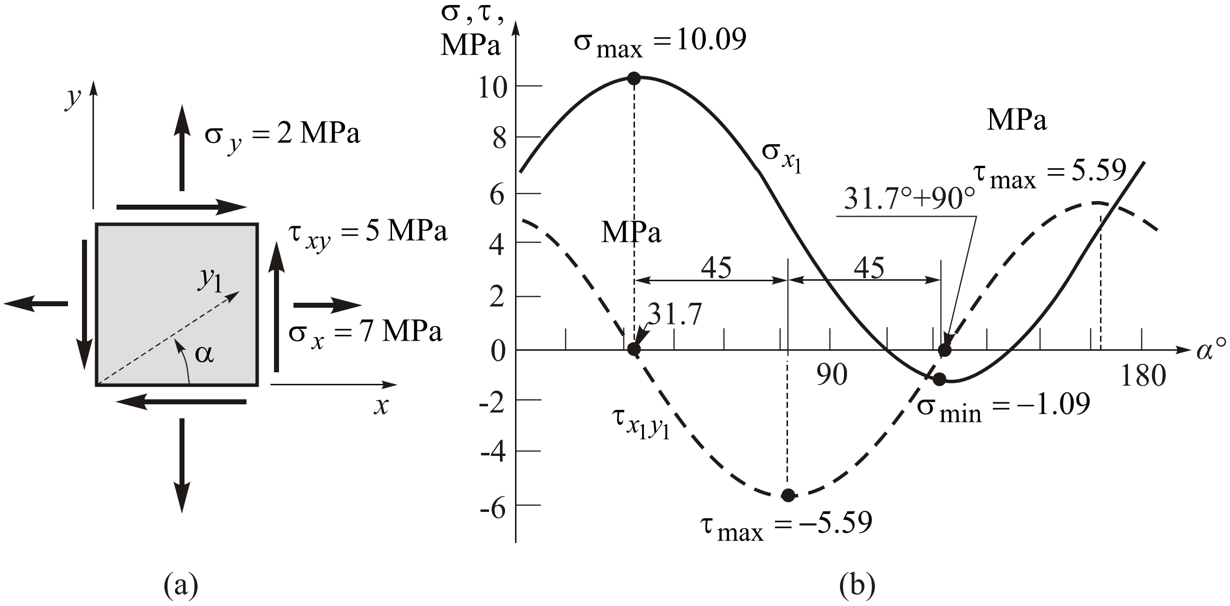
and

$$\sigma_{y_1} = \frac{1}{2}(10-5) - \frac{1}{2}(10+5)\cos 240° + 6\sin 240° = 1.05\,\text{MPa}.$$

The results are indicated in Fig. c.

Example 2

A two-dimensional stress state at a point in a loaded structure is shown in Fig. a. (1) Write the stress-transformation equations. (2) Compute σ_{x_1} and $\tau_{x_1 y_1}$ with α between 0 and 180° in 15° increments for $\sigma_x = 7\,$MPa, $\sigma_y = 2\,$MPa, and $\tau_{xy} = 5$ MPa. Plot the graphs $\sigma_{x_1}(\alpha)$ and $\tau_{x_1 y_1}(\alpha)$.



Variation in normal stress σ_{x_1} and shearing stress $\tau_{x_1 y_1}$ with angle α varying between 0 and 180°

Solution (1) We express Eqs. (9) and (10) as follows:

$$\sigma_{x_1} = A + B\cos 2\alpha + C\sin 2\alpha,$$
$$\tau_{x_1 y_1} = -B\sin 2\alpha + C\cos 2\alpha,$$

where

$$A = \frac{1}{2}\left(\sigma_x + \sigma_y\right), \quad B = \frac{1}{2}\left(\sigma_x - \sigma_y\right), \quad C = \tau_{xy}.$$

(2) Substitution of the prescribed values into Eqs. (9) and (10) results in

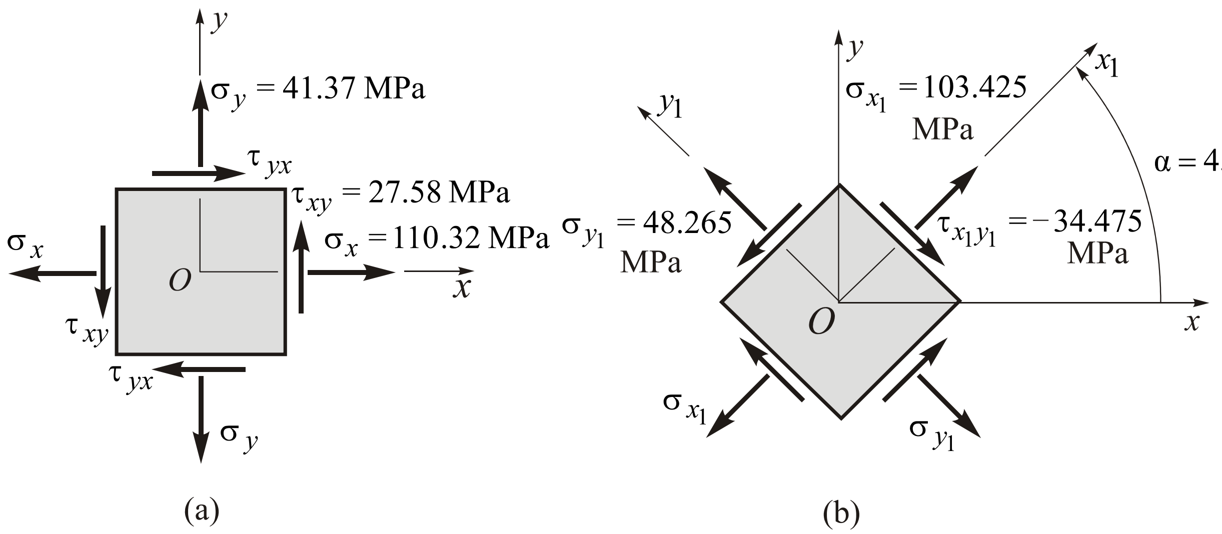
$$\sigma_{x_1} = 4.5 + 2.5\cos 2\alpha + 5\sin 2\alpha,$$

$$\tau_{x_1 y_1} = -2.5\sin 2\alpha + 5\cos 2\alpha.$$

Here, permitting α to vary from 0 to 180° in increments of 15° yields the data upon which the curves shown in Fig. b are based. These cartesian representations indicate how the stresses vary around a point. Observe that the ***direction of maximum (and minimum) shear stress bisects the angle between the maximum and minimum normal stresses***. Moreover, the normal stress is either a maximum or a minimum on planes $\alpha = 31.7°$ and $\alpha = 31.7° + 90°$, respectively, for which the shearing stress is zero. **Note. The conclusions drawn from the foregoing are valid for any state of stress**.

Example 3

An element in plane stress is subjected to stresses $\sigma_x = 110.32$ MPa, $\sigma_y = 41.37$ MPa, and $\tau_{xy} = \tau_{yx} = 27.58$ MPa, as shown in Fig. a. Determine the stresses acting on an element inclined at an angle $\alpha = 45°$.



(a) Element in plane stress, and (b) element inclined at an angle $\alpha = 45°$

Solution To determine the stresses acting on an inclined element, we will use the transformation equations (Eqs. (9) and (10)). From the given numerical data, we obtain the following values for substitution into those equations:

$$\frac{\sigma_x + \sigma_y}{2} = 75.845 \text{ MPa}, \quad \frac{\sigma_x - \sigma_y}{2} = 34.475 \text{ MPa}, \quad \tau_{xy} = 27.58 \text{ MPa},$$

$$\sin 2\alpha = \sin 90° = 1, \qquad \cos 2\alpha = \cos 90° = 0.$$

Substituting these values into Eqs. (9) and (10), we get

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha =$$

$$= 75.845 \text{ MPa} + (34.475 \text{ MPa})(0) + (27.58 \text{ MPa})(1) = 103.425 \text{ MPa},$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha =$$

$$= -(34.475\text{ MPa})(1) + (27.58\text{ MPa})(0) = -34.475\text{ MPa}.$$

In addition, the stress σ_{y_1} may be obtained from Eq. (11):

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha =$$

$$= 75.845\text{ MPa} - (34.475\text{ MPa})(0) - (27.58\text{ MPa})(1) = 48.265\text{ MPa}.$$

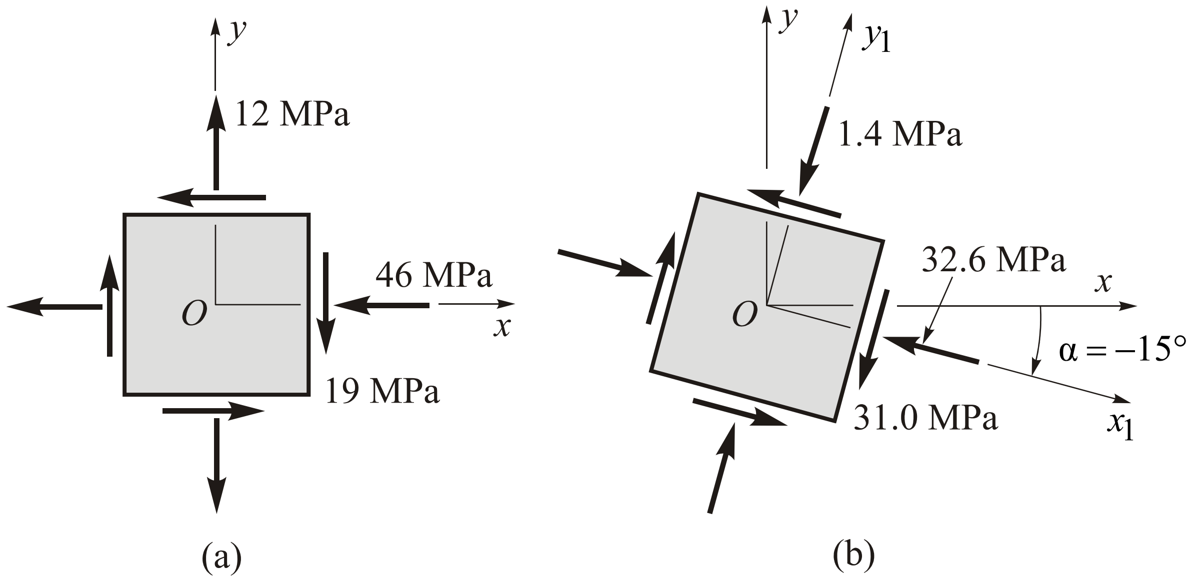
From these results we can obtain the stresses acting on all sides of an element oriented at $\alpha = 45°$, as shown in Fig. b. The arrows show the true directions in which the stresses act. Note especially the directions of the shear stresses, all of which have the same magnitude. Also, observe that the sum of the normal stresses remains constant and equal to 151.69 MPa from Eq. (12):

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y = 151.69\text{ MPa}.$$

Note. The stresses shown in Fig. b represent the same intrinsic state of stress as do the stresses shown in Fig. a. However, the stresses have different values because the elements on which they act have different orientations.

Example 4

On the surface of a loaded structure a plane stress state exists at a point, where the stresses have the magnitudes and directions shown on the stress element of Fig. a. Determine the stresses acting on an element that is oriented at a clockwise angle of 15° with respect to the original element.



Solution The stresses acting on the original element (see Fig. a) have the following values:

$$\sigma_x = -46\,\text{MPa}, \quad \sigma_y = 12\,\text{MPa}, \quad \tau_{xy} = -19\,\text{MPa}.$$

An element oriented at a clockwise angle of 15° is shown in Fig. b, where the x_1 axis is at an angle $\alpha = -15°$ with respect to the *x* axis (clockwised rotation).

We will calculate the stresses on the x_1 face of the element oriented at $\alpha = -15°$ by using the transformation equations (Eqs. (9) and (10)). The components are:

$$A = \frac{\sigma_x + \sigma_y}{2} = -17\,\text{MPa}, \quad B = \frac{\sigma_x - \sigma_y}{2} = -29\,\text{MPa},$$

$$\sin 2\alpha = \sin(-30°) = -0.5, \quad \cos 2\alpha = \cos(-30°) = 0.8660\,.$$

Substituting into the transformation equations, we get

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha =$$

$$= -17\text{ MPa} + (-29\text{ MPa})(0.8661) + (-19\text{ MPa})(-0.5) = -32.6\text{ MPa},$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha =$$

$$= -(-29\text{ MPa})(-0.5) + (-19\text{ MPa})(0.8660) = -31.0\text{ MPa}.$$

Also, the normal stress acting on the y_1 face (Eq. (3.10)) is

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha =$$

$$= -17\text{ MPa} - (-29\text{ MPa})(0.8661) - (-19\text{ MPa})(-0.5) = -1.4\text{ MPa}.$$

To check the results, we note that $\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y$.

The stresses acting on the inclined element are shown in Fig. b, where the arrows indicate the true directions of the stresses.

Note. Both stress elements shown in the figure represent the same state of stress.

4 Principal Stresses and Maximum Shear Stresses

The transformation equations for plane stress show that the normal stresses σ_{y_1} and the shear stresses $\tau_{y_1 z_1}$ vary continuously as the axes are rotated through the angle α. This variation is pictured in Fig. 10 for a particular combination of stresses. From the figure, we see that both the normal and shear stresses reach maximum and minimum values at 90° intervals. These ***maximum and minimum values are usually needed for design purposes***. For instance, fatigue failures of structures such as machines and aircraft are often associated with the maximum stresses, and hence their magnitudes and orientations should be determined as part of the design process.

The determination of principal stresses is an example of a type of mathematical analysis known as **eigenvalue problem** in matrix algebra. The stress-transformation equations and the concept of principal stresses are due to the French mathematicians A. L. Cauchy (1789–1857) and Barre de Saint-Venant (1797–1886) and to the Scottish scientist and engineer W. J. M. Rankine (1820–1872).

4.1 Principal Stresses

The maximum and minimum normal stresses, called the **principal stresses**, can be found from the transformation equation for the normal stress σ_{y_1} (Eq. 9). By taking the derivative of σ_{y_1} with respect to α and setting it equal to zero, we obtain an equation from which we can find the values of at which σ_{y_1} is a maximum or minimum. The equation for the derivative is

$$\frac{d\sigma_{y_1}}{d\alpha} = -\left(\sigma_y - \sigma_z\right)\sin 2\alpha + 2\tau_{yz}\cos 2\alpha = 0, \tag{21}$$

from which we get

$$\tan 2\alpha_p = \frac{2\tau_{yz}}{\sigma_y - \sigma_z}, \tag{22}$$

or in more simple designation,

$$\tan 2\alpha_p = \frac{2\tau_\alpha}{\sigma_\alpha - \sigma_\beta}, \tag{23}$$

The subscript p indicates that the angle α_p defines the orientation of the **principal planes**, i.e. the planes, on which the principal stresses act.

Two values of the angle $2\alpha_p$ in the range from 0 to 360° can be obtained from Eq. (22). These values differ by 180°, with one value between 0 and 180° and the other between 180° and 360°. Therefore, the angle α_p has two values that differ by 90°, one value between 0 and 90° and the other between 90° and 180°. The two values of α_p are known as the **principal angles**. For one of these angles, the normal stress σ_{y_1} is a **maximum principal stress**; for the other, it is a **minimum principal stress**. Because the principal angles differ by 90°, we see that ***the principal stresses occur on mutually perpendicular planes***.

The principal stresses can be calculated by substituting each of the two values of α_p into the first stress-transformation equation (Eq. 9) and solving for σ_{y_1}. By determining the principal stresses in this manner, we not only obtain the values of the principal stresses but we also learn ***which principal stress is associated with which principal angle***.

Let us obtain the formulas for the principal stresses, using right triangle in Fig. 15, constructed from Eq. (22). The hypotenuse of the triangle, obtained from the Pythagorean theorem, is

$$R = \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2}\,. \tag{24}$$

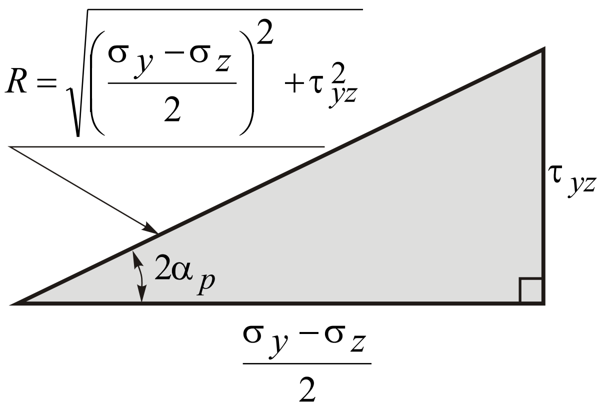


Fig. 15 Geometric analogue of Eq. (22)

The quantity R is always a positive number and, like the other two sides of the triangle, has units of stress. From the triangle we obtain two additional relations:

$$\cos 2\alpha_p = \frac{\sigma_y - \sigma_z}{2R}, \tag{25}$$

$$\sin 2\alpha_p = \frac{\tau_{yz}}{R}. \tag{26}$$

Now we substitute these expressions for $\cos 2\alpha_p$ and $\sin 2\alpha_p$ into Eq. (9) and obtain the **algebraically larger of the two principal stresses**, denoted by σ_1:

$$\sigma_1 = \frac{\sigma_y + \sigma_z}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2}\,. \tag{27}$$

The smaller of the principal stresses, denoted by σ_3, may be found from the condition that the sum of the normal stresses on perpendicular planes is constant (see Eq. 12):

$$\sigma_1 + \sigma_3 = \sigma_y + \sigma_z\,. \tag{28}$$

Substituting the expression for σ_1 into Eq. (28) and solving for σ_3, we get

$$\sigma_3 = \frac{\sigma_y + \sigma_z}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2}\,. \tag{29}$$

The formulas for σ_1 and σ_3 can be combined into a single ***formula for the principal stresses:***

$$\sigma_{\max,\min} = \sigma_{1,3} = \frac{\sigma_y + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2}\,. \tag{30}$$

Note. The plus sign gives the algebraically larger principal stress and the minus sign gives the algebraically smaller principal stress.

Let us now find two angles defining the principal planes as α_{p_1} and α_{p_2}, corresponding to the principal stresses σ_1 and σ_3, respectively. Both angles can be determines from the equation for $\tan 2\alpha_p$ (Eq. 22). To correlate the principal angles and principal stresses we will use Eqs. (25) and (26) to find α_p since the only angle that satisfies **both** of those equations is α_{p_1}. Thus, we can rewrite those equations as follows:

$$\cos 2\alpha_{p_1} = \frac{\sigma_y - \sigma_z}{2R}, \tag{31}$$

$$\sin 2\alpha_{p_1} = \frac{\tau_{yz}}{R}. \tag{32}$$

Only one angle exists between 0 and 360° that satisfies both of these equations. Thus, the value of α_{p_1} can be determined uniquely from Eqs. (31) and (32). The angle α_{p_2}, corresponding to σ_3, defines a plane that is perpendicular to the plane defined by α_{p_1}. Therefore, α_{p_2} can be taken as 90° larger or 90° smaller than α_{p_1}.

It is very important to evaluate the value of shear stresses acting at principal planes since it was noted earlier that they are zero. For this purpose, we will use the transformation equation for the shear stresses (Eq. (10)). If we set the shear stress $\tau_{y_1 z_1}$ equal to zero, we get an equation that is the same as Eq. (21). It means that the ***angles to the planes of zero shear stress are the same as the angles to the principal planes***. Thus, ***the shear stresses are zero on the principal planes***.

The principal planes for elements in **uniaxial stress** and **biaxial stress** are the *y* and *z* planes themselves (Fig. 16), because $\tan 2\alpha_p = 0$ (see Eq. 22) and the two values of α_p are 0 and 90°. We also know that the *y* and *z* planes are the principal planes from the fact that the shear stresses are zero on those planes.

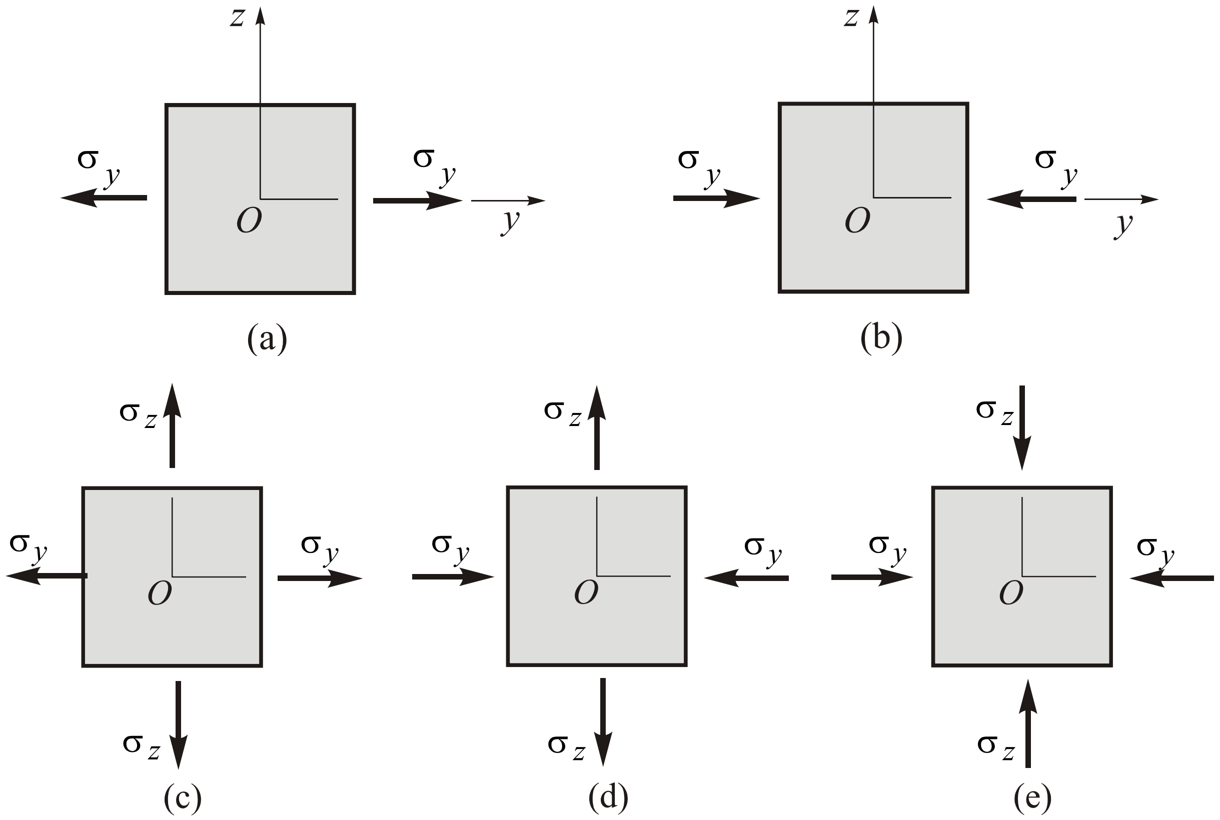


Fig. 16 Elements in uniaxial (a) and (b) and biaxial (c), (d), (e) stress state:

(a) $\sigma_y = 80\text{ MPa} = \sigma_1$, $\sigma_z = 0 = \sigma_{2(3)}$, $\sigma_x = 0 = \sigma_{3(2)}$;

(b) $\sigma_y = -80\text{ MPa} = \sigma_3$, $\sigma_z = 0 = \sigma_{1(2)}$, $\sigma_x = 0 = \sigma_{2(1)}$;

(c) $\sigma_y = 60\text{ MPa} = \sigma_1$, $\sigma_z = 25\text{ MPa} = \sigma_2$, $\sigma_x = 0 = \sigma_3$;

(d) $\sigma_y = -60\text{ MPa} = \sigma_3$, $\sigma_z = 25\text{ MPa} = \sigma_1$, $\sigma_x = 0 = \sigma_2$;

(e) $\sigma_y = -60\text{ MPa} = \sigma_3$, $\sigma_z = -25\text{ MPa} = \sigma_2$, $\sigma_x = 0 = \sigma_1$

For an element in **pure shear** (Fig. 17a), the principal planes are oriented at 45° to the y axis (Fig. 17b), because $\tan 2\alpha_p$ is infinite and the two values of α_p are 45° and 135°. If τ_{yz} is positive, the principal stresses are $\sigma_1 = \tau_{yz}$ and $\sigma_3 = -\tau_{yz}$.

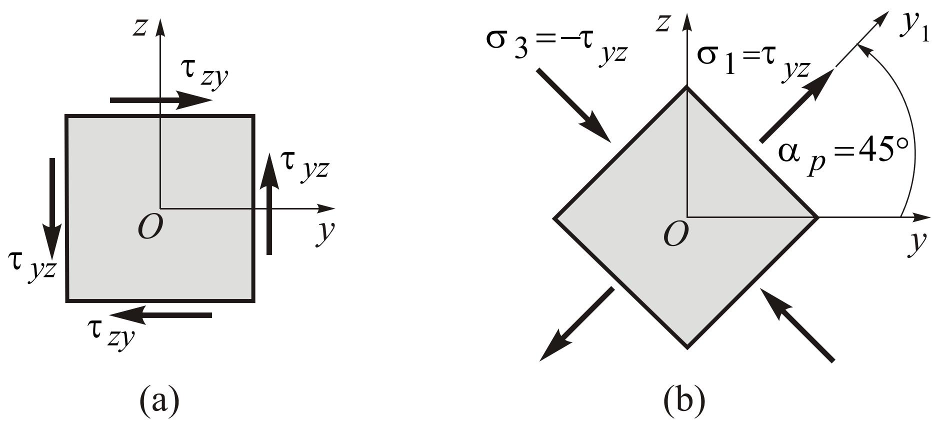


Fig. 17 Element in pure shear

Two principal stresses determined from Eq. (30) are called the **in-plane principal stresses**, since they refer only to rotation of axes in the *zy* plane, that is rotation about the *x* axis. Really any stress element is three-dimensional (Fig. 18a) and has three (not two) principal stresses acting on three mutually perpendicular planes. By making a more complete three-dimensional analysis, it can be shown that the three principal planes for a plane-stress element are the two principal planes already described plus the *x* face of the element. These principal planes are shown in Fig. 18b, where a stress element has been oriented at the principal angle θ_{p_1} which corresponds to the principal stress σ_1. The principal stresses σ_1 and σ_2 are given by Eq. (30), and the third principal stress (σ_3) equals zero. By definition, ***σ_1 is algebraically the largest and σ_3 is algebraically the smallest one***.

Note. There are no shear stresses on any of the principal planes.

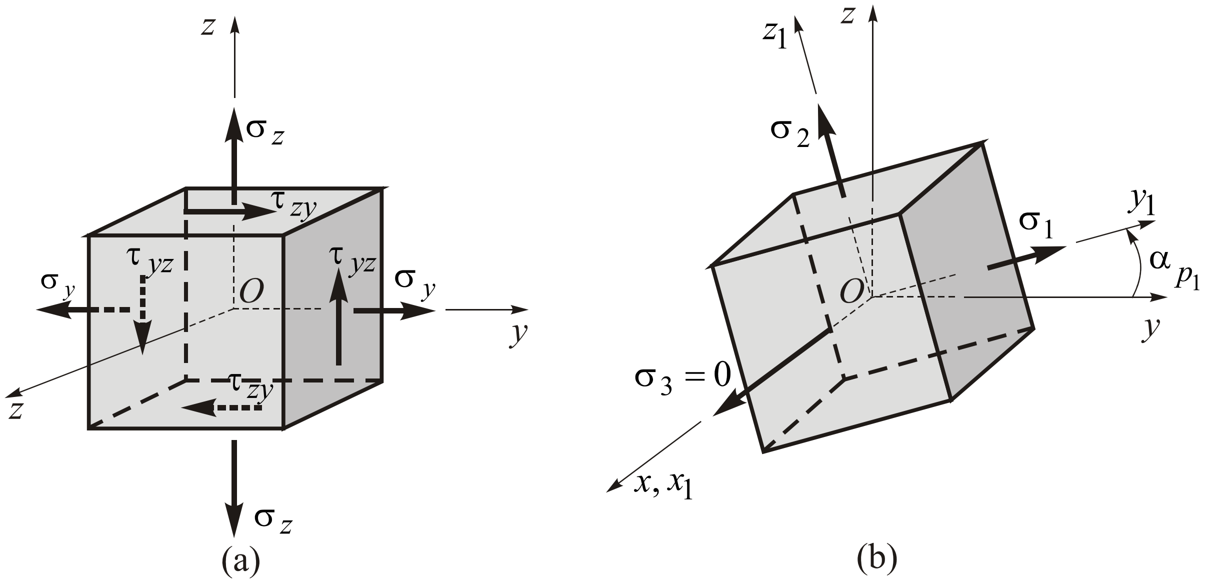


Fig. 18 Elements in plane stress: (a) original element, and (b) element oriented to the three principal planes and three principal stresses

4.2 Maximum Shear Stresses

Now we consider the determination of the maximum shear stresses and the planes on which they act. The shear stresses $\tau_{y_1 z_1}$ acting on inclined planes are given by the second transformation equation (Eq. 10). Equating the derivative of $\tau_{y_1 z_1}$ with respect to α to zero, we obtain

$$\frac{d\tau_{y_1 z_1}}{d\alpha} = -\left(\sigma_y - \sigma_z\right)\cos 2\alpha - 2\tau_{yz}\sin 2\alpha = 0\,, \tag{33}$$

from which

$$\tan 2\alpha_s = -\frac{\sigma_y - \sigma_z}{2\tau_{yz}}\,. \tag{34}$$

The subscript *s* indicates that the angle α_s defines the orientation of the planes of maximum positive and negative shear stresses. Equation (34) yields one value of α_s between 0 and 90° and another between 90° and 180°. These two values differ by 90°, and therefore the maximum shear stresses occur on perpendicular planes. Because shear stresses on perpendicular planes are equal in absolute value, the maximum positive and negative shear stresses differ only in sign.

Comparing Eq. (34) for α_s with Eq. (22) for α_p shows that

$$\tan 2\alpha_s = -\frac{1}{\tan 2\alpha_p} = -\cot 2\alpha_p\,. \tag{35}$$

This equation is the relationship between the angles α_s, and α_p. Let us rewrite this equation in the form

$$\frac{\sin 2\alpha_s}{\cos 2\alpha_s} + \frac{\cos 2\alpha_p}{\sin 2\alpha_p} = 0\,, \tag{36}$$

or

$$\sin 2\alpha_s \sin 2\alpha_p + \cos 2\alpha_s \cos 2\alpha_p = 0\,. \tag{37}$$

Eq. (37) is equivalent to the following expression:

$$\cos\left(2\alpha_s - 2\alpha_p\right) = 0\,.$$

Therefore,

$$2\alpha_s - 2\alpha_p = \pm 90°,$$

and

$$\alpha_s = \alpha_p \pm 45°. \tag{38}$$

Note. Eq. (38) shows that the planes of maximum shear stress occur at 45° to the principal planes.

The plane of the maximum positive shear stress $\tau_{\max}$ is defined by the angle α_{s_1}, for which the following equations apply:

$$\cos 2\alpha_{s_1} = \frac{\tau_{yz}}{R}, \tag{39}$$

$$\sin 2\alpha_{s_1} = -\frac{\sigma_y - \sigma_z}{2R}, \tag{40}$$

in which R is given by Eq. (24). Also, the angle α_{s_1} is related to the angle α_{p_1} (see Eqs. (31) and (32)) as follows:

$$\alpha_{s_1} = \alpha_{p_1} - 45°. \tag{41}$$

Corresponding maximum shear stress is obtained by substituting the expressions for $\cos 2\alpha_{s_1}$ and $\sin 2\alpha_{s_1}$ into the second transformation equation (Eq. 8), yielding

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2}. \tag{42}$$

The maximum negative shear stress has the same magnitude but opposite sign.

Another expression for the maximum shear stress $\tau_{\max}$ can be obtained from the principal stresses σ_1 and σ_3, both of which are given by Eq. (30). Subtracting the expression for σ_3 from that for σ_1 and then comparing with Eq. (42), we see that

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}. \tag{43}$$

Note. Maximum shear stress is equal to one-half the difference of the principal stresses.

The planes of maximum shear stress $\tau_{\max}$ also contain normal stresses. The normal stress acting on the planes of maximum positive shear stress can be determined by substituting the expressions for the angle α_{s_1} (Eqs. (39) and (40)) into the equation for σ_{y_1} (Eq. 9). The resulting stress is equal to the **average** of the normal stresses on the y and z planes:

$$\sigma_{aver} = \frac{\sigma_y + \sigma_z}{2}. \tag{44}$$

This same normal stress acts on the planes of maximum negative shear stress.

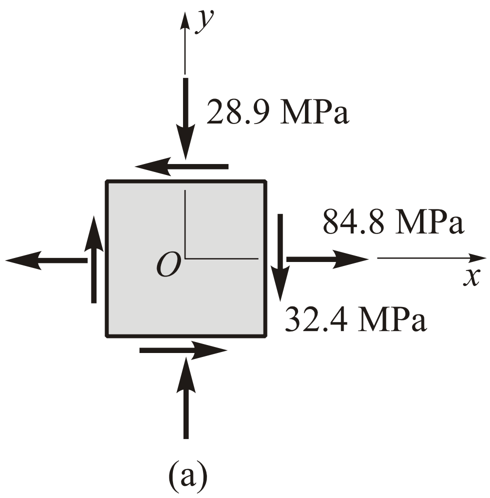
In the particular cases of uniaxial stress and biaxial stress (Fig. 16), the planes of maximum shear stress occur at 45° to the y and z axes. In the case of pure shear (Fig. 17), the maximum shear stresses occur on the y and z planes.

The analysis of shear stresses has dealt only with the stresses acting in the yz plane, i.e. in-plane shear stress. The maximum in-plane shear stresses were found on an element obtained by rotating the x, y, z axes (Fig. 18a) about the x_1 axis through an angle of 45° to the principal planes. The principal planes for the element of Fig. 18a are shown in Fig. 18b.

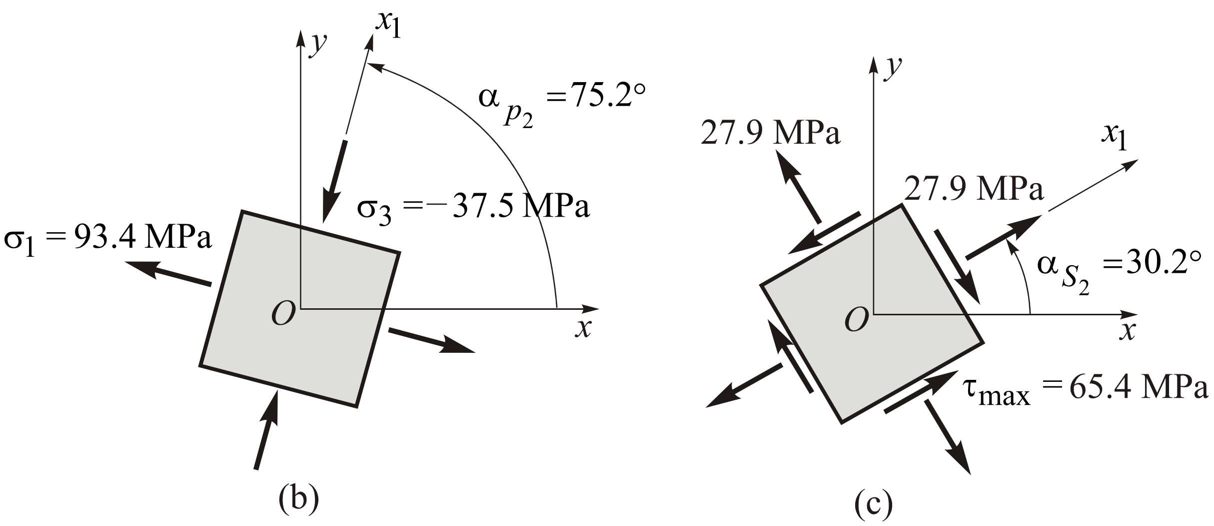
We can also obtain maximum shear stresses by 45° rotations about the other two principal axes (the y_1 and z_1 axes in Fig. 18b). As a result, we obtain three sets of maximum positive and maximum negative shear stresses (compare with Eq. (43)).

Example 5

An element in plane stress is subjected to stresses $\sigma_x = 84.8\,\text{MPa}$, $\sigma_y = -28.9\,\text{MPa}$, and $\tau_{xy} = -32.4\,\text{MPa}$, as shown in Fig. a. (1) Determine the principal stresses and show them on a sketch of a properly oriented element; (2) Determine the maximum shear stresses and show them on a sketch of a properly oriented element.



(a) Element in plane stress



(b) principal stresses; and (c) maximum shear stresses

Solution (1) ***Calculation of principal stresses***. The principal angles α_p that locate the principal planes can be obtained from Eq. (22):

$$\tan 2\alpha_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-32.4 \text{ MPa})}{84.8 \text{ MPa} - (-28.9 \text{ MPa})} = -0.5697.$$

Solving for the angles, we get the following two sets of values:

$$2\alpha_p = 150.3° \quad \text{and} \quad \alpha_p = 75.2°,$$

$$2\alpha_p = 330.3° \quad \text{and} \quad \alpha_p = 165.2°.$$

The principal stresses may be obtained by substituting the two values of $2\alpha_p$ into the transformation equation for σ_{x_1} (Eq. (9)). Determine preliminary the following quantities:

$$A = \frac{\sigma_x + \sigma_y}{2} = \frac{84.8 \text{ MPa} - 28.9 \text{ MPa}}{2} = 27.9 \text{ MPa},$$

$$B = \frac{\sigma_x - \sigma_y}{2} = \frac{84.8 \text{ MPa} + 28.9 \text{ MPa}}{2} = 56.8 \text{ MPa}.$$

Now we substitute the first value of $2\alpha_p$ into Eq. (9) and obtain

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha =$$

$$= 27.9 \text{ MPa} + (56.8 \text{ MPa})(\cos 150.3°) - (32.4 \text{ MPa})(\sin 150.3°) = -37.5 \text{ MPa}.$$

By the similar way, we substitute the second value of $2\alpha_p$ and obtain $\sigma_{x_1} = 93.4$ MPa. In result, the principal stresses and their corresponding principal angles are

$$\sigma_1 = 93.4 \text{ MPa} \text{ and } \alpha_{p_1} = 165.2°$$

$$\sigma_3 = -37.5 \text{ MPa} \text{ and } \alpha_{p_2} = 75.2°.$$

Keep in mind, that $\sigma_2 = 0$ acts in *z* direction.

Note, that α_{p_1} and α_{p_2} differ by 90° and that $\sigma_1 + \sigma_3 = \sigma_x + \sigma_y$.

The principal stresses are shown on a properly oriented element in the Fig. b. Of course, the principal planes are free from shear stresses.

The principal stresses may also be calculated directly from Eq. (30):

$$\sigma_{1,2(3)} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} =$$

$$= 27.9 \text{ MPa} \pm \sqrt{(56.8 \text{ MPa})^2 + (-32.4 \text{ MPa})^2},$$

$$\sigma_{1,2(3)} = 27.9 \text{ MPa} \pm 65.4 \text{ MPa}.$$

Therefore,

$$\sigma_1 = 93.4 \text{ MPa}, \ \sigma_3 = -37.5 \text{ MPa}, \ (\sigma_2 = 0).$$

(2) ***Maximum shear stresses***. The maximum in-plane shear stresses are given by Eq. (42):

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(56.8 \text{ MPa})^2 + (-32.4 \text{ MPa})^2} = 65.4 \text{ MPa}.$$

The angle α_{s_1} to the plane having the maximum positive shear stress is calculated from Eq. (41):

$$\alpha_{s_1} = \alpha_{p_1} - 45° = 165.2° - 45° = 120.2°.$$

It follows that the maximum negative shear stress acts on the plane for which $\alpha_{s_2} = 120.2° - 90° = 30.2°$.

The normal stresses acting on the planes of maximum shear stresses are calculated from Eq. (44):

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = 27.9 \text{ MPa}.$$

Finally, the maximum shear stresses and associated normal stresses are shown on the stress element of Fig. c.

5 Circular Diagrams for Plane Stress (Mohr's circles)

The basic equations of stress transformation derived earlier may be interpreted graphically. The graphical technique permits the rapid transformation of stress from one plane to another and also provides an overview of the state of stress at a point. It provides a means for calculating principal stresses, maximum shear stresses, and stresses on inclined planes. This method was devised by the German civil engineer Otto Christian Mohr (1835–1918), who developed a plot known as **Mohr's circle** in 1882. Mohr's circle is valid not only for stresses, but also for other quantities of a similar nature, including strains and moments of inertia.

5.1 Equation of Mohr's circle

The equations of Mohr's circle can be derived from the transformation equations for plane stress (Eqs. ((9), (10)). These two equations may be represented as

$$\sigma_{y_1} - \frac{\sigma_y + \sigma_z}{2} = \frac{\sigma_y - \sigma_z}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha , \tag{45}$$

$$\tau_{y_1 z_1} = -\frac{\sigma_y - \sigma_z}{2}\sin 2\alpha + \tau_{yz}\cos 2\alpha . \tag{46}$$

Squaring each equation, adding them, and simplifying, we obtain well-known **equation of a circle**:

$$\left(\sigma_{y_1} - \frac{\sigma_y + \sigma_z}{2}\right)^2 + \tau_{y_1 z_1}{}^2 = \left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2 . \tag{47}$$

This equation can be written in more simple form using the following notation:

$$\sigma_{\text{aver}} = \frac{\sigma_y + \sigma_z}{2} . \tag{48}$$

$$R = \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2} . \tag{49}$$

Equation (47) now becomes

$$\left(\sigma_{y_1} - \sigma_{\text{aver}}\right)^2 + \tau_{y_1 z_1}{}^2 = R^2 , \tag{50}$$

which is the equation of a circle in standard algebraic form. The coordinates are σ_{y_1} and $\tau_{y_1 z_1}$, the radius is R and the center of the circle has coordinates $\sigma_{y_1} = \sigma_{\text{aver}}$ and $\tau_{y_1 z_1} = 0$.

5.2 Mohr's circle construction

Mohr's circle can be plotted from Eqs. (45, 46) and (50) in two different ways. We will plot the normal stress σ_{y_1} positive to the right and the shear stress $\tau_{y_1 z_1}$ positive downward, as shown in Fig. 19. ***The advantage of plotting shear stresses***

positive downward is that the angle 2α on Mohr's circle is positive when counterclockwise, which agrees with the positive direction of 2α in the derivation of the transformation equations.

Mohr's circle can be constructed in a variety of ways, depending upon which stresses are known and which are unknown. Let us assume that we know the stresses σ_y, σ_z and τ_{yz} acting on the *y* and *z* planes of an element in plane stress (Fig. 20a). This information is sufficient to construct the circle. Then, with the circle drawn, we can determine the stresses σ_y, σ_z and $\tau_{y_1 z_1}$ acting on an inclined element (Fig. 20b). We can also obtain the principal stresses and maximum shear stresses from the circle.

With σ_y, σ_z and τ_{yz} known, the procedure for constructing Mohr's circle is as follows (see Fig. 20c):

(a) Draw a set of coordinate axes with σ_{y_1} as abscissa (positive to the right) and $\tau_{y_1 z_1}$ as ordinate (positive downward).

(b) Locate the center *C* of the circle at the point having coordinates $\sigma_{y_1} = \sigma_{\text{aver}}$ and $\tau_{y_1 z_1} = 0$ (see Eqs. (48) and (50)).

(c) Locate point *A*, representing the stress conditions on the *y* face of the element shown in Fig. 20a, by plotting its coordinates $\sigma_{y_1} = \sigma_y$ and $\tau_{y_1 z_1} = \tau_{yz}$. Note that point *A* corresponds to $\alpha = 0$. The *y* face of the element (Fig. 20a) is labeled "*A*" to show its correspondence with point *A* in the diagram.

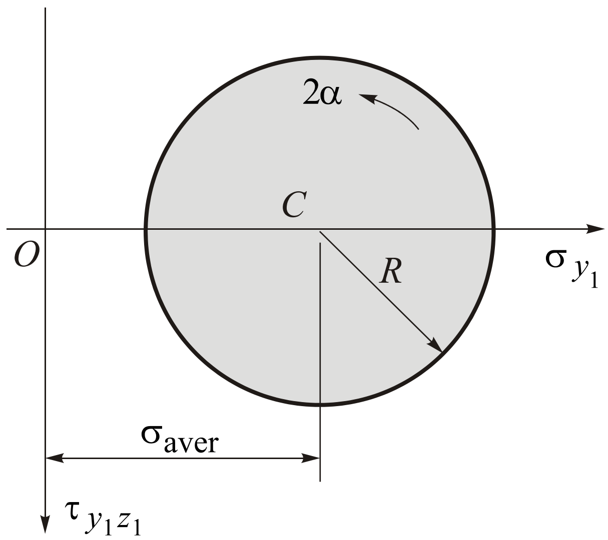


Fig 19 The form of Mohr's circle with $\tau_{y_1 z_1}$ positive downward and the angle 2α positive counterclockwise

(d) Locate point *B* representing the stress conditions on the *z* face of the element shown in Fig. 20a, by plotting its coordinates $\sigma_{y_1} = \sigma_z$ and $\tau_{y_1 z_1} = -\tau_{yz}$. Point *B* corresponds to $\alpha = 90°$. The *z* face of the element (Fig. 20a) is labeled "*B*" to show its correspondence with point *B* in the diagram.

(e) Draw a line from point *A* to point *B*. It is a diameter of the circle and passes through the center *C*. Points *A* and *B*, representing the stresses on planes at $90°$ to each other, are at opposite ends of the diameter (and therefore are $180°$ apart on the circle).

(f) Using point C as the center, draw Mohr's circle though points A and B. The circle drawn in this manner has radius R (Eq. (49)).

Note. When Mohr's circle is plotted to scale, numerical results can be obtained graphically.

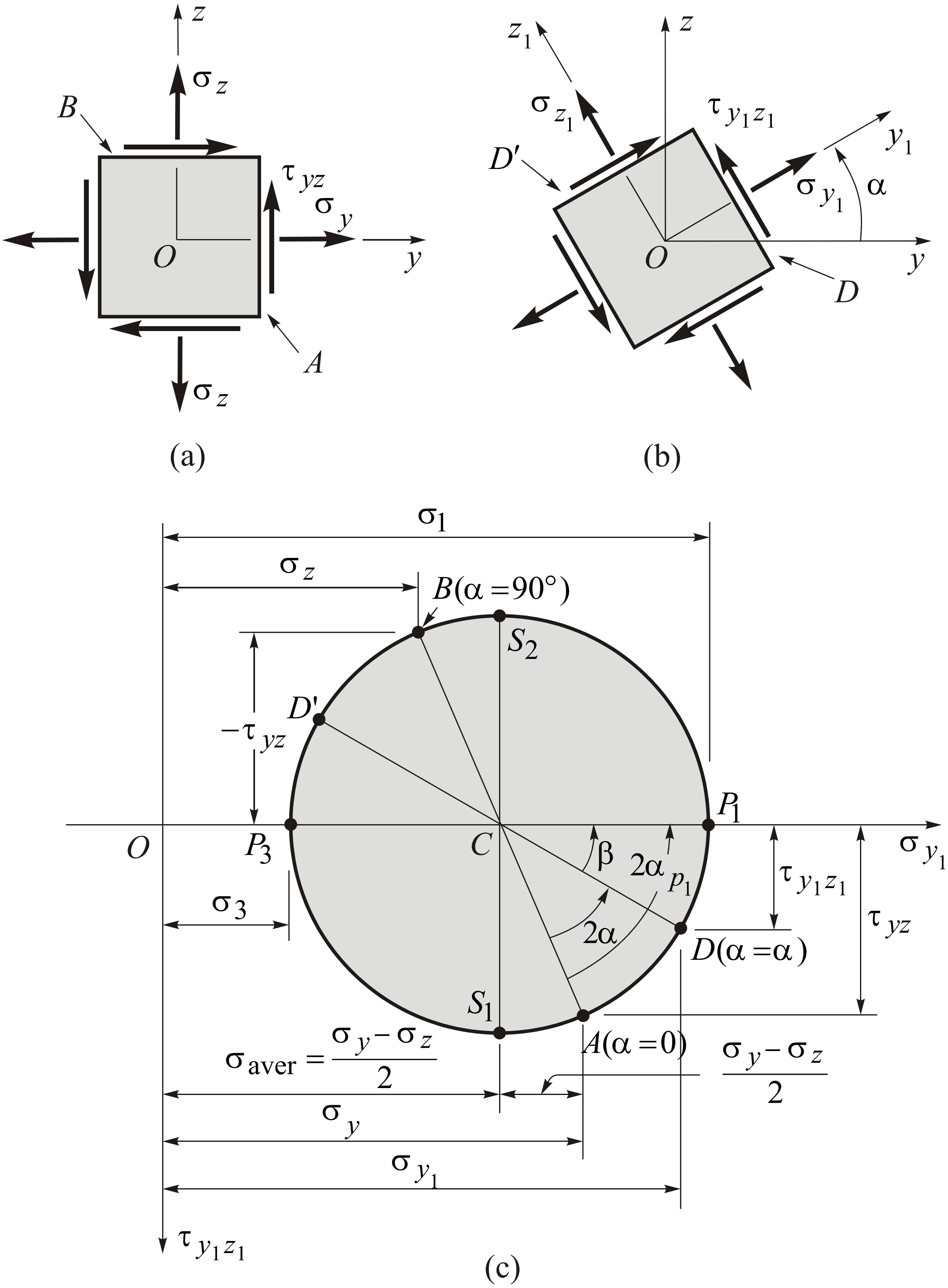


Fig. 20 Construction of Mohr's circle for plane stress

(1) ***Stresses on an inclined element***. Mohr's circle shows how the stresses represented by points on it are related to the stresses acting on an element. The stresses on an inclined plane defined by the angle α (Fig. 20b) are found on the circle at the point where the angle from the reference point (point *A*) is 2α. Thus, as we rotate the $y_1 z_1$ axes counterclockwise through an angle α (Fig. 20b), the point on Mohr's circle corresponding to the y_1 face moves counterclockwise through an angle 2α. Similarly, in clockwise rotation of the axes, the point on the circle moves clockwise through an angle twice as large.

(2) ***Principal stresses***. The determination of principal stresses is the most important application of Mohr's circle. As we move around Mohr's circle (Fig. 20c), we encounter point P_1 where the normal stress reaches its algebraically largest value and the shear stress is zero. Hence, point P_1 gives the algebraically larger principal stress and its angle $2\alpha_{p_1}$ from the reference point *A* ($\alpha = 0$) gives the orientation of the principal plane. The next principal plane, associated with the algebraically smallest normal stress, is represented by point P_3, diametrically opposite to point P_1.

(3) ***Maximum shear stresses***. Points S_1 and S_2 which represent the planes of maximum positive and maximum negative shear stresses, respectively, are located at the bottom and top of Mohr's circle (Fig. 20c). These points are at angles $2\alpha = 90°$ from points P_1 and P_3, which agrees with the fact that the planes of maximum shear stress are oriented at $45°$ to the principal planes. The maximum shear stresses are numerically equal to the radius *R* of the circle. Also, the normal stresses on the planes of maximum shear stress are equal to the abscissa of point *C*, which is the average normal stress σ_{aver}.

Various multiaxial states of stress can readily be treated by applying the foregoing procedure. Fig. 21 shows some examples of Mohr's circles for commonly encountered cases. Analysis of material behavior subject to different loading conditions is often facilitated by this type of compilation. Interestingly, for the case of equal tension and compression (this type of stress state was named as pure shear) (see Fig. 21a), $\sigma_x = 0$ and the *x*-directed strain does not exist ($\varepsilon_x = 0$). Hence the element is in a state of ***plane strain as well as plane stress***. An element in this condition can be converted to a condition of pure shear by rotating it 45° as indicated.

In the case of triaxial tension (Fig. 21b and 22a), a Mohr's circle is drawn corresponding to each projection of a three-dimensional element (see Fig. 22b). The three-circle cluster represents Mohr's circle for triaxial stress. The case of tension with lateral pressure (Fig. 21c) is explained similarly.

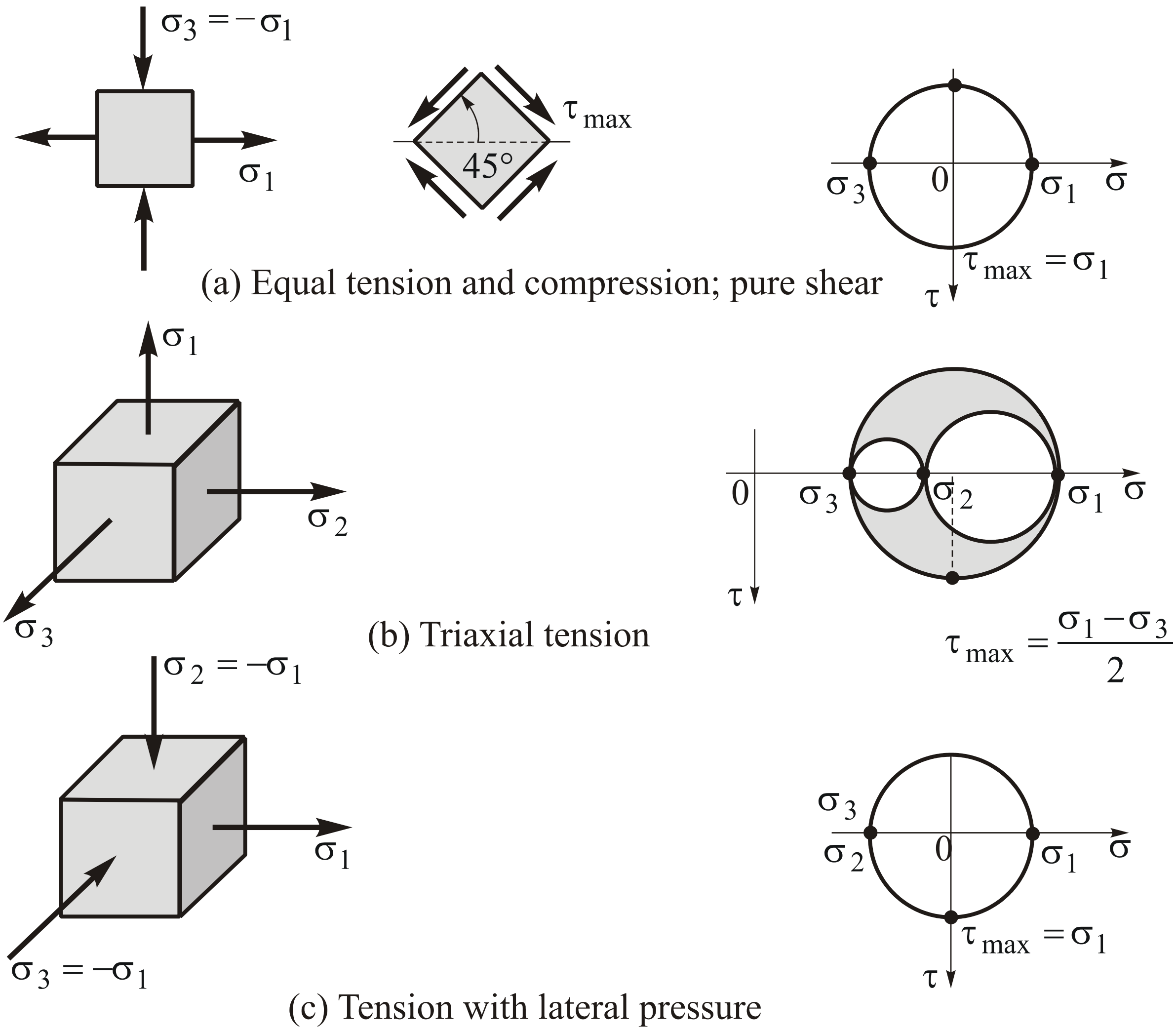


Fig. 21 Mohr's circle for various states of stress

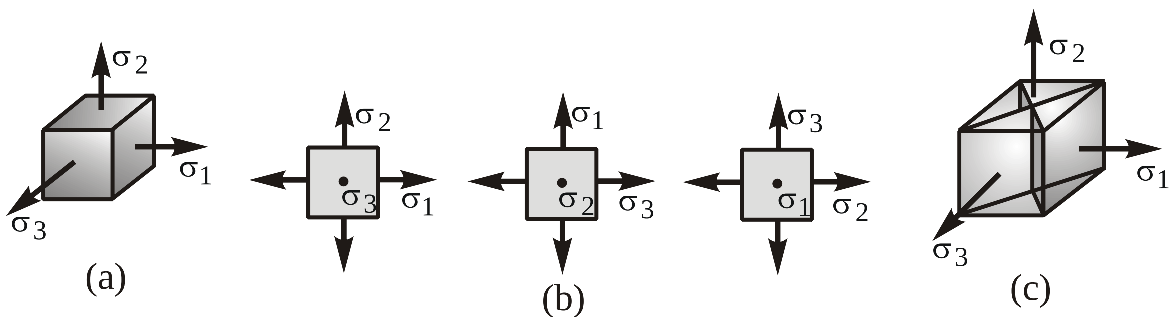
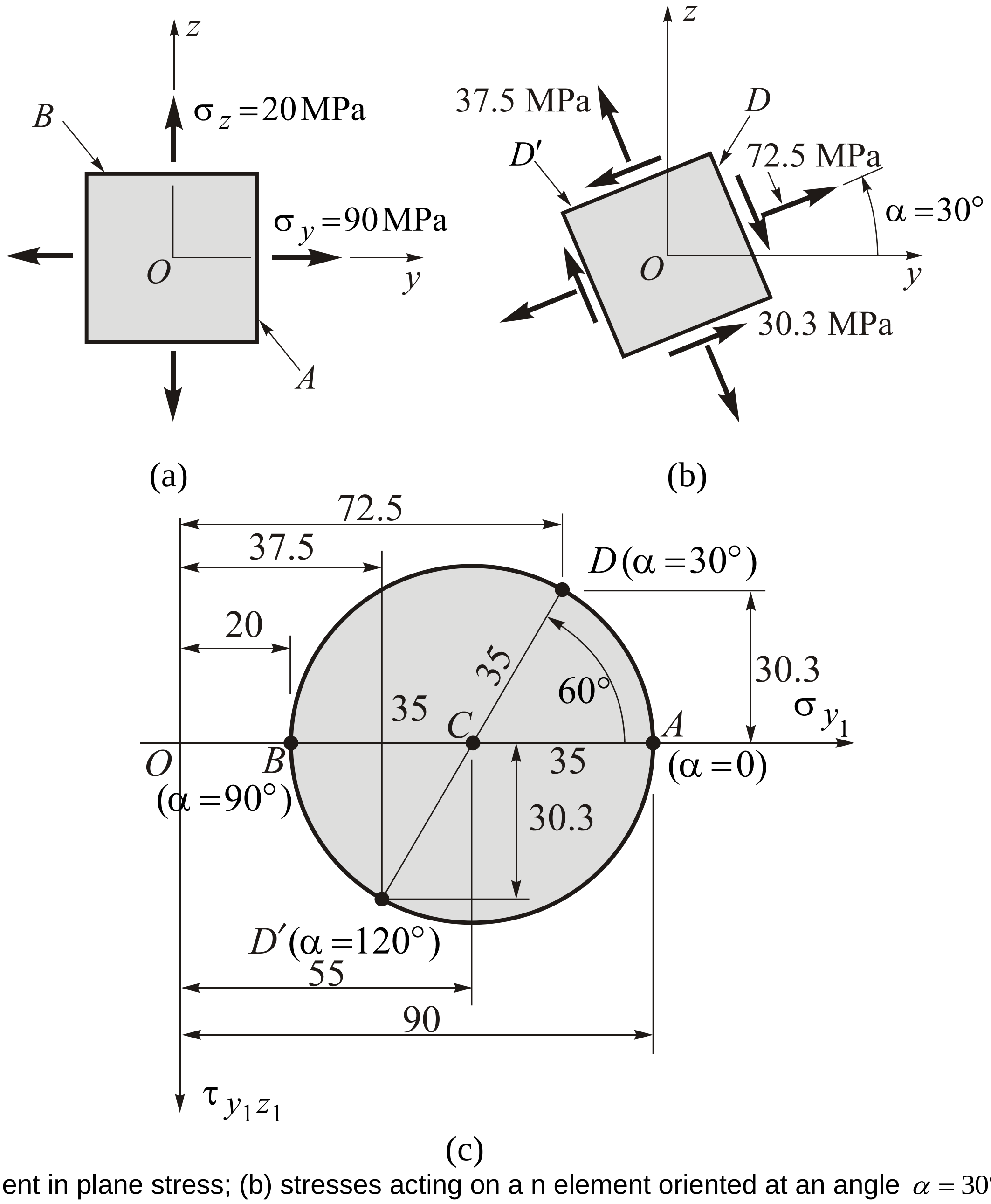


Fig. 22 Three-dimensional state of stress

Note. Mohr's circle eliminates the need to remember the formulas of stress transformation.

Example 6

At a point on the surface of a cylinder, loaded by internal pressure, the material is subjected to biaxial stresses $\sigma_y = 90\,\text{MPa}$ and $\sigma_z = 20\,\text{MPa}$, as shown on the stress element of figure (a). Using Mohr's circle, determine the stresses acting on an element inclined at an angle $\alpha = 30°$. (Consider only the in-plane stresses, and show the results on a sketch of a properly oriented element).



(c)

(a) Element in plane stress; (b) stresses acting on a n element oriented at an angle $\alpha = 30°$; (c) the corresponding Mohr's circle (Note: All stresses on the circle have units of MPa)

Solution (1) ***Construction of Mohr's circle.*** Let us set up the axes for the normal and shear stresses, with σ_{y_1} positive to the right and $\tau_{y_1 z_1}$ positive downward, as shown in figure (c). Then we place the center *C* of the circle on the σ_{y_1} axis at the point where the stress equals the average normal stress:

$$\sigma_{\text{aver}} = \frac{\sigma_y + \sigma_z}{2} = \frac{90\ \text{MPa} + 20\ \text{MPa}}{2} = 55\ \text{MPa}.$$

Point *A*, representing the stresses on the *y* face of the element ($\alpha = 0$), has coordinates

$$\sigma_{y_1} = 90\ \text{MPa}\,,\ \tau_{y_1 z_1} = 0\,.$$

Similarly, the coordinates of point *B*, representing the stresses on the *z* face ($\alpha = 90°$), are

$$\sigma_{y_1} = 20\ \text{MPa}\,,\ \tau_{y_1 z_1} = 0\,.$$

Now we draw the circle through points *A* and *B* with center at *C* and radius *R* equal to

$$R = \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^{\ 2}} = \sqrt{\left(\frac{90\ \text{MPa} - 20\ \text{MPa}}{2}\right)^2 + 0} = 35\ \text{MPa}.$$

(2) ***Stresses on an element inclined at*** $\boldsymbol{\alpha = 30°}$***.*** The stresses acting on a plane oriented at an angle $\alpha = 30°$ are given by the coordinates of point *D*, which is at an angle $2\alpha = 60°$ from point *A* (see figure (c)). By inspection of the circle, we see that the coordinates of point *D* are

$$\sigma_{y_1} = \sigma_{\text{aver}} + R\cos 60° = 55\ \text{MPa} + (35\ \text{MPa})(\cos 60°) = 72.5\ \text{MPa},$$

$$\tau_{y_1 z_1} = -R\cos 60° = -(35\ \text{MPa})(\cos 60°) = -30.3\ \text{MPa}.$$

In a similar manner, we can find the stresses represented by point D', which corresponds to an angle $\alpha = 120°$ (or $2\alpha = 240°$):

$$\sigma_{y_1} = \sigma_{\text{aver}} - R\cos 60° = 55\ \text{MPa} - (35\ \text{MPa})(\cos 60°) = 37.5\ \text{MPa},$$

$$\tau_{y_1 z_1} = R\cos 60° = (35\ \text{MPa})(\cos 60°) = 30.3\ \text{MPa}.$$

These results are shown in figure (b) on a sketch of an element oriented at an angle $\alpha = 30°$, with all stresses shown in their true directions.

Note. The sum of the normal stresses on the inclined element is equal to $\sigma_y + \sigma_z$ or 110 MPa.

Example 7

An element in plane stress at the surface of a structure is subjected to stresses $\sigma_y = 100$ MPa, $\sigma_z = 35$ MPa, and $\tau_{yz} = 30$ MPa, as shown in figure (a). Using Mohr's circle, determine the following quantities: (1) the stresses acting on an element inclined at an angle $\alpha = 40°$, (2) the principal stresses, and (3) the maximum shear stresses. Consider only the in-plane stresses, and show all results on sketches of properly oriented elements.

Solution (1) ***Construction of Mohr's circle.*** Let us set up the axes for Mohr's circle, with σ_{y_1} positive to the right and $\tau_{y_1 z_1}$ positive downward (see figure (c)). The center *C* of the circle is located on the σ_{y_1} axis at the point where σ_{y_1} equals the average normal stress:

$$\sigma_{\text{aver}} = \frac{\sigma_y + \sigma_z}{2} = \frac{100 \text{ MPa} + 35 \text{ MPa}}{2} = 67.5 \text{ MPa}.$$

Point *A*, representing the stresses on the *y* face of the element ($\alpha = 0$), has coordinates

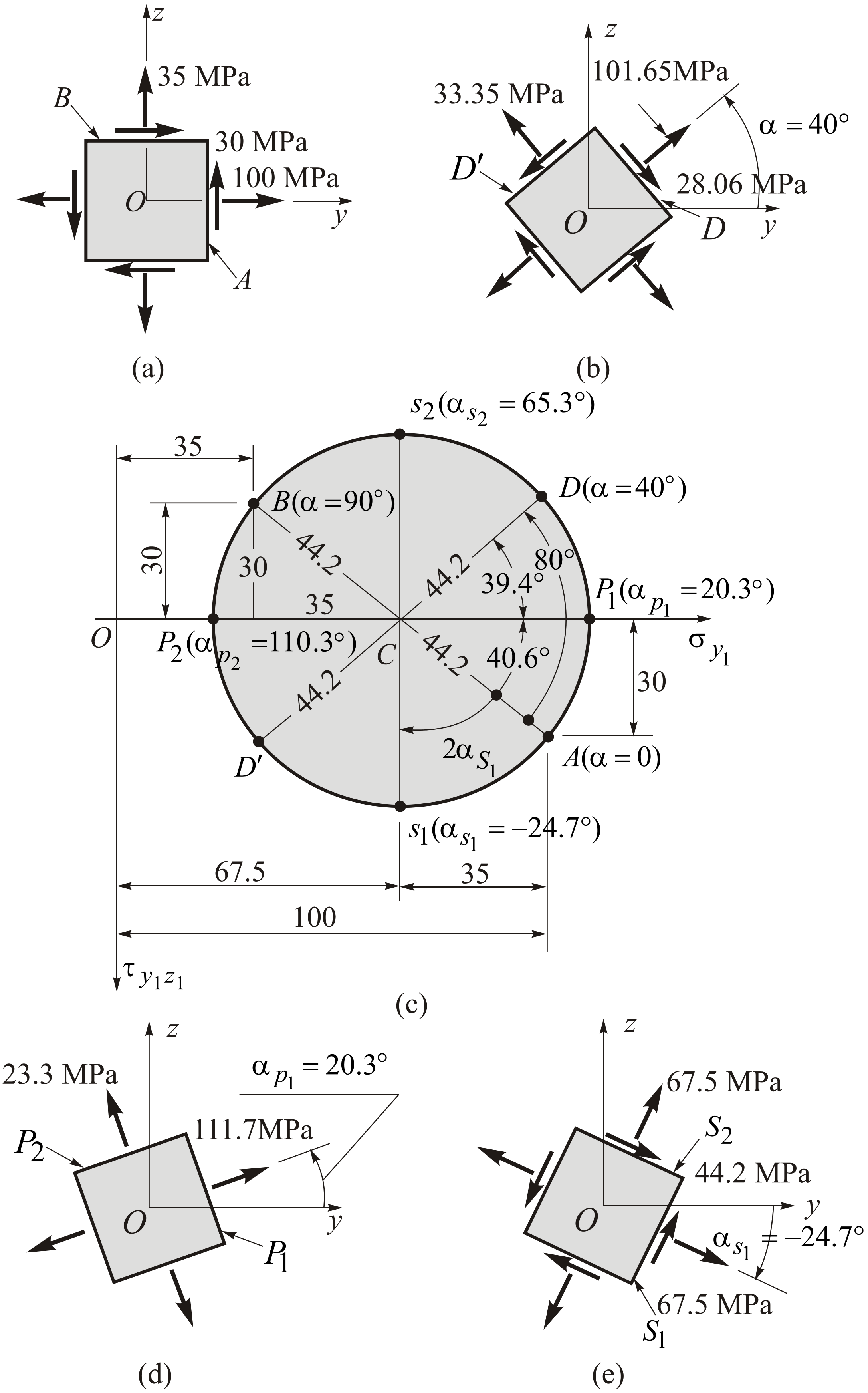
$$\sigma_{y_1} = 100 \text{MPa}, \quad \tau_{y_1 z_1} = 30 \text{MPa}.$$

Similarly, the coordinates of point *B*, representing the stresses on the *z* face ($\alpha = 90$), are

$$\sigma_{y_1} = 35 \text{MPa}, \quad \tau_{x_1 y_1} = -30 \text{MPa}.$$

The circle is now drawn through points *A* and *B* with center at *C*. The radius of the circle is

$$R = \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}{}^2} = \sqrt{\left(\frac{100 \text{ MPa} - 35 \text{ MPa}}{2}\right)^2 + (30 \text{ MPa})^2} = 44.2 \text{ MPa}.$$



(a) Element in plane stress; (b) stress acting on an element oriented at $\alpha = 40°$; (c) the corresponding Mohr's circle; (d) principal stresses; (e) maximum shear stresses

(2) ***The stresses acting on a plane oriented at an angle*** $\boldsymbol{\alpha = 40°}$**.** They are given by the coordinates of point *D*, which is at an angle $2\alpha = 80°$ from point *A* (see figure (c)). To calculate these coordinates, we need to know the angle between line *CD* and the σ_{y_1} axis (that is, angle DCP_1), which in turn requires that we know the angle between line *CA* and the σ_{y_1} axis (angle ACP_1). These angles are found from the geometry of the circle, as follows:

$$\tan \overline{ACP_1} = \frac{30 \text{ MPa}}{35 \text{ MPa}} = 0.857, \quad \overline{ACP_1} = 40.6°,$$

$$\overline{DCP_1} = 80° - \overline{ACP_1} = 80° - 40.6° = 39.4°.$$

Knowing these angles, we can determine the coordinates of point *D* directly from the figure:

$$\sigma_{y_1} = 67.5 \text{ MPa} + (44.2 \text{ MPa})(\cos 39.4°) = 101.65 \text{ MPa},$$

$$\tau_{y_1 z_1} = -(44.2 \text{ MPa})(\sin 39.4°) = -28.06 \text{ MPa}.$$

In an analogous manner, we can find the stresses represented by point D', which corresponds to a plane inclined at an angle $\alpha = 130°$ (or $2\alpha = 260°$):

$$\sigma_{y_1} = 67.5 \text{ MPa} - (44.2 \text{ MPa})(\cos 39.4°) = +33.35 \text{ MPa},$$

$$\tau_{y_1 z_1} = (44.2 \text{ MPa})(\sin 39.4°) = 28.06 \text{ MPa}.$$

These stresses are shown in figure (c) on a sketch of an element oriented at an angle $\alpha = 40°$ (all stresses are shown in their true directions).

Note. The sum of the normal stresses is equal to $\boldsymbol{\sigma_x + \sigma_y}$ or 135 MPa.

(3) ***Principal stresses.*** The principal stresses are represented by points P_1 and P_2 on Mohr's circle (see figure (c)). The algebraically larger principal stress (point P_1) is

$$\sigma_1 = 67.5 \text{ MPa} + 44.2 \text{ MPa} = 111.7 \text{ MPa},$$

as seen by inspection of the circle. The angle $2\alpha_{p_1}$ to point P_1 from point *A* is the angle ACP_1 on the circle, that is,

$$\overline{ACP_1} = 2\alpha_{p_1} = 40.6°,\ \alpha_{p_1} = 20.3°.$$

Thus, the plane of the algebraically larger principal stress is oriented at an angle $\alpha_{p_1} = 20.3°$, as shown in figure (d).

The algebraically smaller principal stress (represented by point P_2) is obtained from the circle in a similar manner:

$$\sigma_2 = 67.5\ \text{MPa} - 44.2\ \text{MPa} = 23.3\ \text{MPa}.$$

The angle $2\alpha_{p_2}$ to point P_2 on the circle is $40.6° + 180° = 220.6°$; thus, the second principal plane is defined by the angle $\alpha_{p_2} = 110.3°$. The principal stresses and principal planes are shown in the figure (d).

Note. The sum of the normal stresses is equal to 135 MPa.

(4) ***Maximum shear stresses.*** The maximum shear stresses are represented by points S_1 and S_2 on Mohr's circle; therefore, the maximum in-plane shear stress (equal to the radius of the circle) is

$$\tau_{\max} = 44.2\ \text{MPa}.$$

The angle ACS_1 from point A to point S_1 is $90° - 40.6° = 49.4°$, and therefore the angle $2\alpha_{s_1}$, for point S_1 is

$$2\alpha_{s_1} = -49.4°.$$

This angle is negative because it is measured clockwise on the circle. The corresponding angle α_{s_1} to the plane of the maximum positive shear stress is one-half that value, or $\alpha_{s_1} = -24.7°$, as shown in Figs. (c) and (e). The maximum negative shear stress (point S_2 on the circle) has the same numerical value as the maximum positive stress (44.2 MPa).

The normal stresses acting on the planes of maximum shear stress are equal to σ_{aver}, which is the abscissa of the center C of the circle (67.5 MPa). These stresses are also shown in figure (e).

Note. The planes of maximum shear stresses are oriented at 45° to the principal planes.

Example 8

At a point on the surface of a shaft the stresses are $\sigma_y = -50\,\text{MPa}$, $\sigma_z = 10\,\text{MPa}$, and $\tau_{yz} = -40\,\text{MPa}$, as shown in figure (a). Using Mohr's circle, determine the following quantities: (1) the stresses acting on an element inclined at an angle $\alpha = 45°$, (2) the principal stresses, and (3) the maximum shear stresses.

Solution (1) ***Construction of Mohr's circle***. The axes for the normal and shear stresses in the Mohr's circle are shown in figure (c), with σ_{y_1} positive to the right and $\tau_{y_1 z_1}$ positive downward. The center *C* of the circle is located on the σ_{y_1} axis at the point where the stress equals the average normal stress:

$$\sigma_{\text{aver}} = \frac{\sigma_y + \sigma_z}{2} = \frac{-50\text{ MPa} + 10\text{ MPa}}{2} = -20\text{ MPa}.$$

Point *A*, representing the stresses on the *y* face of the element ($\alpha = 0$), has coordinates

$$\sigma_{y_1} = -50\,\text{MPa}, \quad \tau_{y_1 z_1} = -40\,\text{MPa}.$$

Similarly, the coordinates of point *B*, representing the stresses on the *z* face ($\alpha = 90°$), are

$$\sigma_{y_1} = 10\,\text{MPa}, \quad \tau_{y_1 z_1} = 40\,\text{MPa}.$$

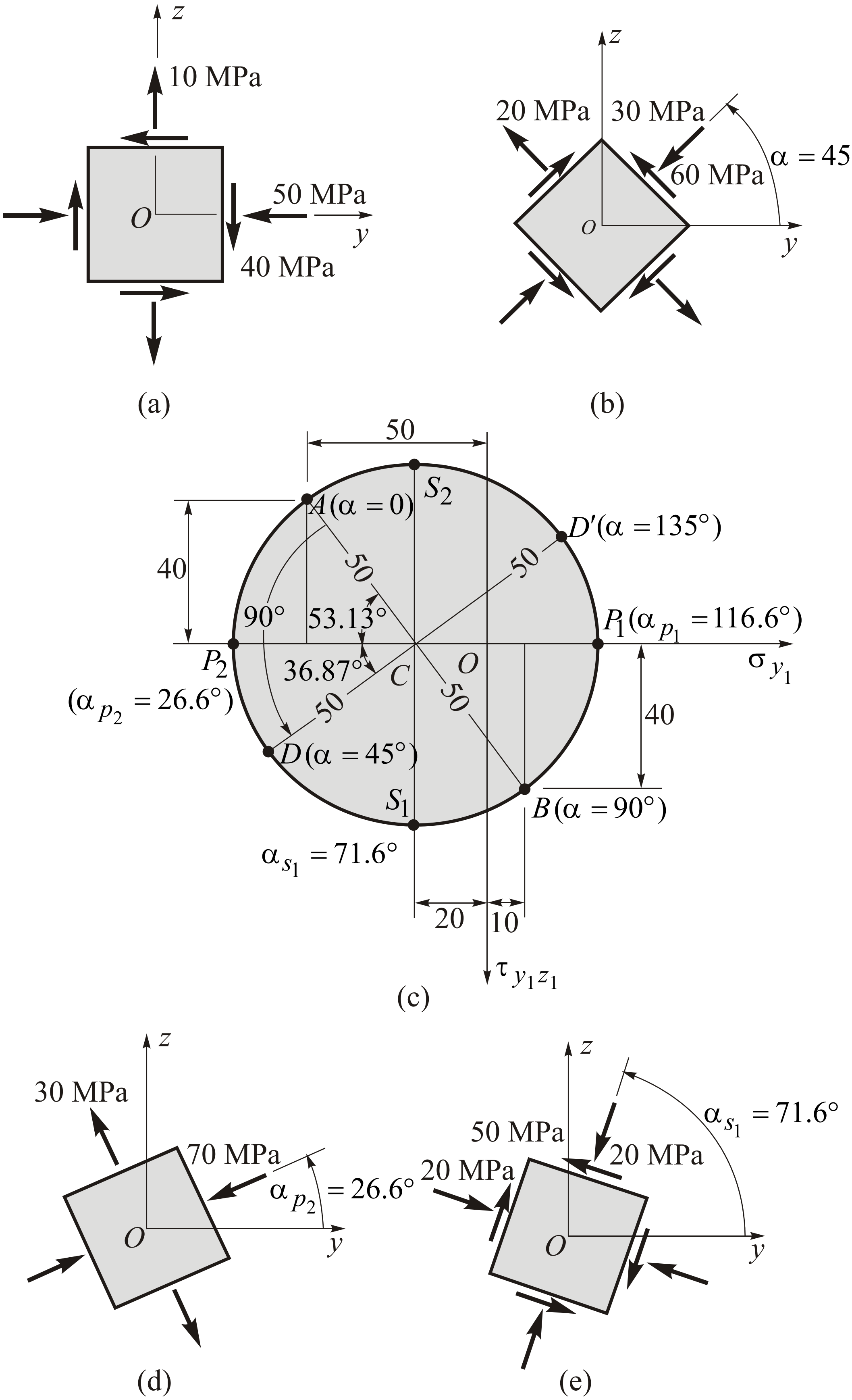
The circle is now drawn through points *A* and *B* with center at *C* and radius *R* equal to:

$$R = \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{xy}^{\;2}} = \sqrt{\left(\frac{-50\text{ MPa} - 10\text{ MPa}}{2}\right)^2 + (-40\text{ MPa})^2} = 50\,\text{MPa}.$$

(2) ***Stresses on an element inclined at*** $\boldsymbol{\alpha = 45°}$. These stresses are given by the coordinates of point *D*, which is at an angle $2\alpha = 90°$ from point *A* (figure (c)). To evaluate these coordinates, we need to know the angle between line *CD* and the negative σ_{y_1} axis (that is, angle DCP_2), which in turn requires that we know the angle between line *CA* and the negative σ_{y_1} axis (angle ACP_2). These angles are found from the geometry of the circle as follows:

$$\tan \overline{ACP_2} = \frac{40\text{ MPa}}{30\text{ MPa}} = \frac{4}{3}, \quad \overline{ACP_2} = 53.13°,$$

$$\overline{DCP_2} = 90° - \overline{ACP_2} = 90° - 53.13° = 36.87°.$$



(a) Element in plane stress; (b) stresses acting on an element oriented at $\alpha = 40°$; (c) the corresponding Mohr's circle; (d) principal stresses, and (e) maximum shear stresses. (Note: All stresses on the circle have units of MPa)

Knowing these angles, we can obtain the coordinates of point D directly from the figure:

$$\sigma_{y_1} = -20\text{ MPa} - (50\text{ MPa})(\cos 36.87°) = -60\text{ MPa},$$

$$\tau_{y_1 z_1} = (50\text{ MPa})(\sin 36.87°) = 30\text{ MPa}.$$

In an analogous manner, we can find the stresses represented by point D', which corresponds to a plane inclined at an angle $\alpha = 135°$ (or $2\alpha = 270°$):

$$\sigma_{y_1} = -20\text{ MPa} + (50\text{ MPa})(\cos 36.87°) = 20\text{ MPa},$$

$$\tau_{y_1 z_1} = (-50\text{ MPa})(\sin 36.87°) = -30\text{ MPa}.$$

These stresses are shown in Fig. b on a sketch of an element oriented at an angle $\alpha = 45°$ (all stresses are shown in their true directions).

Note. The sum of the normal stresses is equal to $\sigma_y + \sigma_z$ or – 40 MPa.

(3) ***Principal stresses***. They are represented by points P_1 and P_2 on Mohr's circle. The algebraically larger principal stress (represented by point P_1) is

$$\sigma_1 = -20\text{ MPa} + 50\text{ MPa} = 30\text{ MPa},$$

as seen by inspection of the circle. The angle $2\alpha_{p_1}$ to point P_1 from point A is the angle ACP_1 measured counterclockwise on the circle, that is,

$$\overline{ACP_1} = 2\alpha_{p_1} = 53.13° + 180° = 233.13°, \quad \alpha_{p_1} = 116.6°.$$

Thus, the plane of the algebraically larger principal stress is oriented at an angle $\alpha_{p_1} = 116.6°$.

The algebraically smaller principal stress (point P_2) is obtained from the circle in a similar manner:

$$\sigma_3 = -20\text{ MPa} - 50\text{ MPa} = -70\text{ MPa}.$$

The angle $2\alpha_{p_2}$ to point P_2 on the circle is 53.13°. The second principal plane is defined by the angle $2\alpha_{p_2} = 26.6°$.

The principal stresses and principal planes are shown in Fig. (d).

Note. The sum of the normal stresses is equal to $\sigma_y + \sigma_z$ or – 40 MPa.

(4) ***Maximum shear stresses***. The maximum positive and negative shear stresses are represented by points S_2 and S_2 on Mohr's circle (figure (c)). Their magnitudes, equal to the radius of the circle, are

$$\tau_{\max} = 50\,\text{MPa}.$$

The angle ACS_1 from point A to point S_1 is $90° + 53.13° = 143.13°$, and therefore the angle $2\alpha_{s_1}$ for point S_1 is

$$2\alpha_{s_1} = 143.13°.$$

The corresponding angle α_{s_1} to the plane of the maximum positive shear stress is one-half that value, or $\alpha_{s_1} = 71.6°$, as shown in figure (e). The maximum negative shear stress (point S_2 on the circle) has the same numerical value as the positive stress (50 MPa).

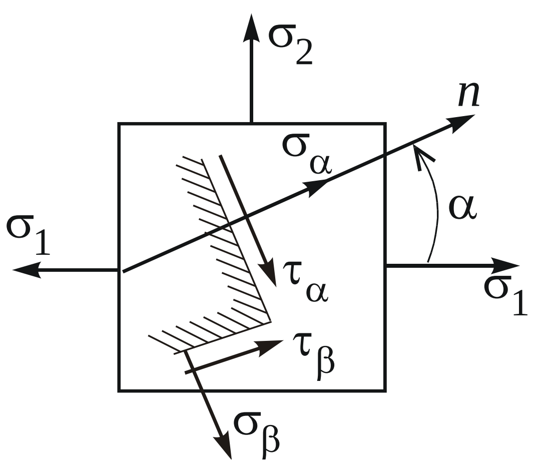
The normal stresses acting on the planes of maximum shear stress are equal to σ_{aver}, which is the coordinate of the center C of the circle ($-20\,\text{MPa}$). These stresses are also shown in figure (e).

Note. The planes of maximum shear stress are oriented at 45° to the principal planes.

6 Examples of Simplified Analytical and Graphical Solutions of the Problems of Plane Stress State

6.1 Direct problem of plane stress state. Determination of stresses on inclined planes

Example 1



Given: $\sigma_1 = 80\,\text{MPa},\ \sigma_2 = 20\,\text{MPa}.$

It is necessary to determine the stresses on the plane of general position with the normal at $\alpha = +30°$ relative to σ_1 direction and also the stresses on perpendicular plane.

Analytical solution

$$\sigma_\alpha = \sigma_1 \cos^2\alpha + \sigma_2 \sin^2\alpha =$$

$$=(+80)\cos^2(+30°)+(+20)\sin^2(+30°)=$$

$$=80\left(\frac{\sqrt{3}}{2}\right)^2+20\left(\frac{1}{2}\right)^2=+65\,\text{MPa},$$

$$\sigma_\beta=\sigma_1\sin^2\alpha+\sigma_2\cos^2\alpha=(+80)\sin^2(+30°)+(+20)\cos^2(+30°)=$$

$$=80\left(\frac{1}{2}\right)^2+20\left(\frac{\sqrt{3}}{2}\right)^2=+35\,\text{MPa}.$$

Checking:

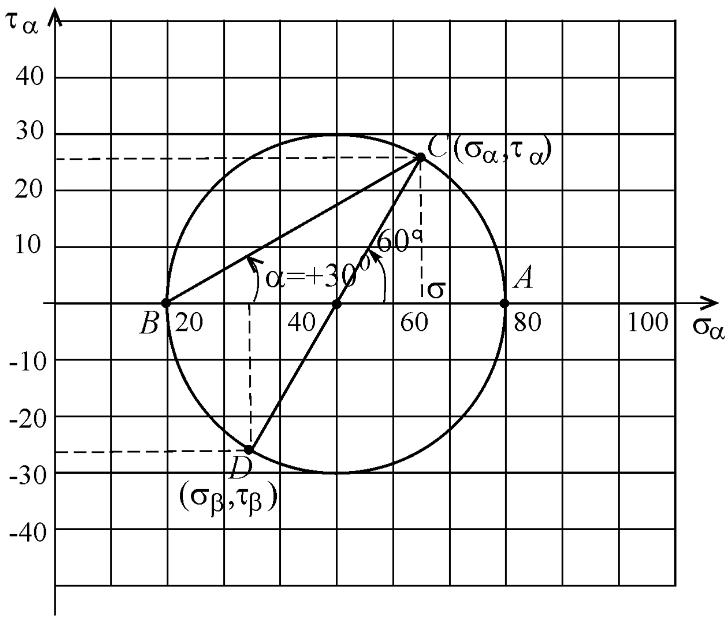
$$\sigma_1+\sigma_2=\sigma_\alpha+\sigma_\beta,$$

$$(80+20=65+35),$$

$$\tau_\alpha=\frac{\sigma_1-\sigma_2}{2}\sin 2\alpha=\frac{(+80)-(+20)}{2}\sin(+60°)=+26\,\text{MPa},$$

$$\tau_\beta=-\tau_\alpha=-26\,\text{MPa}.$$

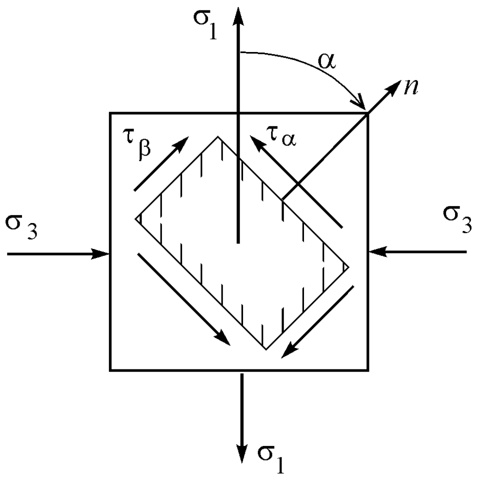
Graphical solution using Mohr's circles.



Given: stresses on the faces of the element are described by two points lying on the diameter of Mohr's circle in system of coordinates (σ,τ): point A $(\sigma_1,0)$, point B $(\sigma_2,0)$.

It is necessary to determine the coordinates of the point C $(\sigma_\alpha,\tau_\alpha)$ and point D $(\sigma_\beta,\tau_\beta)$, which belong to the diameter of the Mohr's circle.

Example 2



Given: Stress state of the element is described by the stresses on two mutually perpendicular planes: $\sigma_1 = 400$ MPa, $\sigma_3 = -400$ MPa.

It is necessary to determine the stresses on the plane of general position with the normal at $\alpha = -45°$ relative to direction of σ_1.

Analytical solution

$$\sigma_\alpha = \sigma_1 \cos^2\alpha + \sigma_3 \sin^2\alpha =$$
$$= (+400)\cos^2(-45°) + (-400)\sin^2(-45°) = 0,$$

$$\sigma_\beta = \sigma_1 \sin^2\alpha + \sigma_3 \cos^2\alpha = (+400)\sin^2(-45°) + (-400)\cos^2(-45°) = 0.$$

Checking:

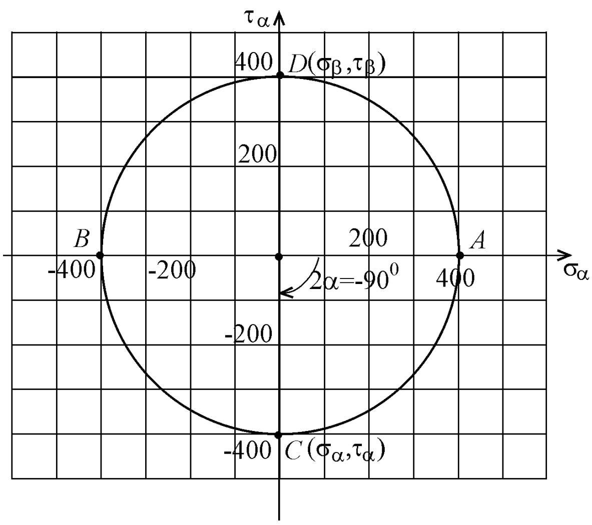
$$\sigma_1 + \sigma_3 = \sigma_\alpha + \sigma_\beta,$$

$$(+400 - 400 = 0 + 0),$$

$$\tau_\alpha = \frac{\sigma_1 - \sigma_3}{2}\sin 2\alpha = \frac{(+400) - (-400)}{2}\sin(-90°) = -400\,\text{MPa},$$

$$\tau_\beta = -\tau_\alpha = +400\,\text{МПа}.$$

Graphical solution using Mohr's circle

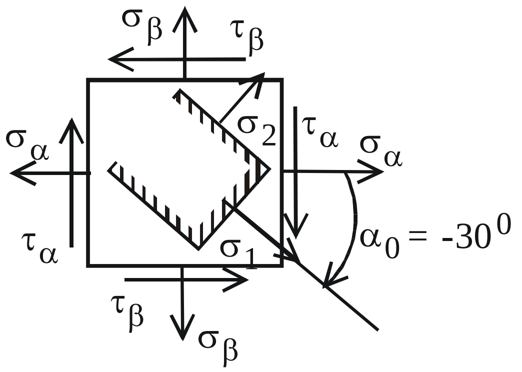


Given: stresses on the faces of the element are represented by two points lying on the diameter of the Mohr's circle: point A $(\sigma_1, 0)$, point B $(\sigma_3, 0)$.

It is necessary to determine coordinates of the point C $(\sigma_\alpha, \tau_\alpha)$ and point D $(\sigma_\beta, \tau_\beta)$, lying on the diameter of Mohr's circle inclined at the angle $2\alpha = -90°$ (clockwise rotation).

6.2 Inverse problem of plain stress state. Determination of principal planes position and values of principal stresses

Example 3



Given:

$$\left.\begin{array}{l}\sigma_\alpha = 65\text{ MPa}\\ \sigma_\beta = 35\text{ MPa}\end{array}\right\},\quad \sigma_\alpha > \sigma_\beta,\quad \tau_\alpha = 26\text{ MPa}.$$

It is necessary to find principal stresses and position of principal planes, i.e.

$\alpha_p - ?$ $\sigma_{\substack{\max\\ \min}} - ?$

Analytical solution

1. Position of principal plane is determined by the angle

$$tg\,2\alpha_0 = \frac{-2\tau_\alpha}{\sigma_\alpha - \sigma_\beta} = -\frac{2(+26)}{(+65)-(+35)} = -\frac{52}{30} = -1.73,$$

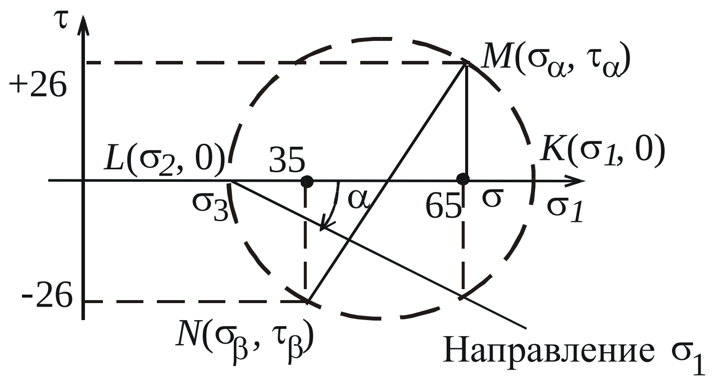
$\alpha_p = -30°$. Note, that α_p should be originated from σ_α direction and to be clockwise.

$$\sigma_{\substack{\max\\ \min}} = \frac{\sigma_\alpha + \sigma_\beta}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_\alpha - \sigma_\beta\right)^2 + 4\tau_\alpha^2} =$$

$$= \frac{(+65)+(+35)}{2} \pm \sqrt{\left((+65)-(+35)\right)^2 + 4(+26)^2} = 50 \pm 30\text{ MPa}$$

$$\sigma_1 = 80\text{ MPa},\quad \sigma_2 = 20\text{ MPa}.$$

Graphical solution



Given:

point M $(\sigma_\alpha, \tau_\alpha)$ and point N $(\sigma_\beta, \tau_\beta)$, which belong to Mohr's circle and are lying on its diameter.

It is necessary to determine position of the point K $(\sigma_1, 0)$ and point L $(\sigma_2, 0)$, also lying on the Mohr's circle and belonging to its diameter. Solution is evident from the Fig.