LECTURE 2

SAFETY FACTOR

LOADS

LOADING CLASS

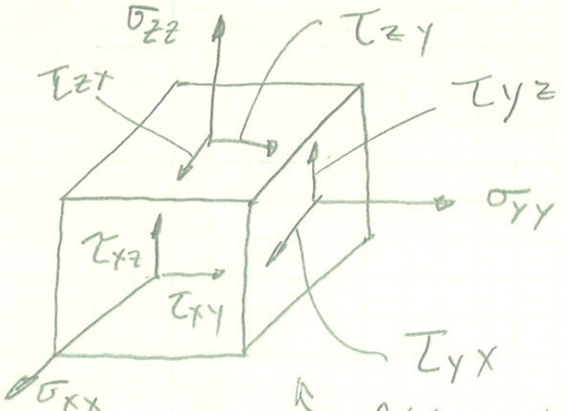
	CONSTANT LOAD	Time Vary Load
STATIONARY ELEMENTS	CLASS 1	CLASS 2
MOVING ELEMENTS	CLASS 3	CLASS 4

The Loads can be determined using Free body diagrams.

4.6 4.7
4.4 4.5

STRESS, STRAIN, AND DEFLECTION

4.1 4.2 4.3



ALL SHOWN AS POSITIVE

We can write a stress tensor

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

σ_{xx} Acts on the x face in the x direction.

We can simplify the normal stresses to

$$\sigma_x \quad \sigma_y \quad \sigma_z$$

Also, the stress tensor is symmetric so:

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$
$$\tau_{zx} = \tau_{xz}$$

For most cases

For a 2D system the tensor is

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

STRAINS

Strains are related by Hook's Law in the elastic region of most materials

$$\sigma = \varepsilon E$$

σ — stress; ε — strain; E — young's modulus

3D Strain Tensor

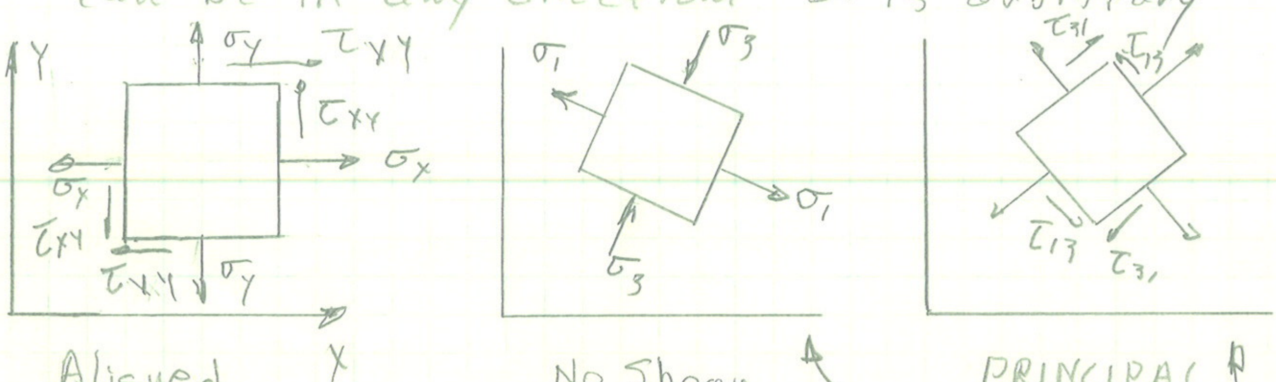
$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

2D Strain Tensor

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}$$

PRINCIPAL STRESSES FOR PLANE STRESS

The orientation of the element of material can be in any direction. It is arbitrary



Aligned With Axes

No Shear Stresses Principal Stresses Only

PRINCIPAL SHEAR

These two are 45° apart

MOHR'S CIRCLE – PLANE STRESS

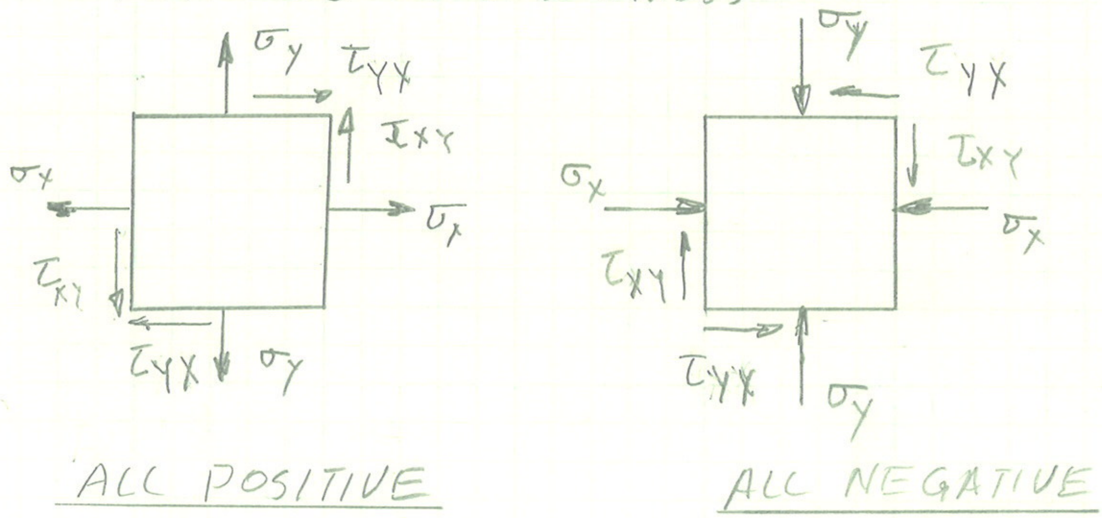
For two dimensional stresses, we can compute the principal stresses and the maximum shear stress using a graphical technique.

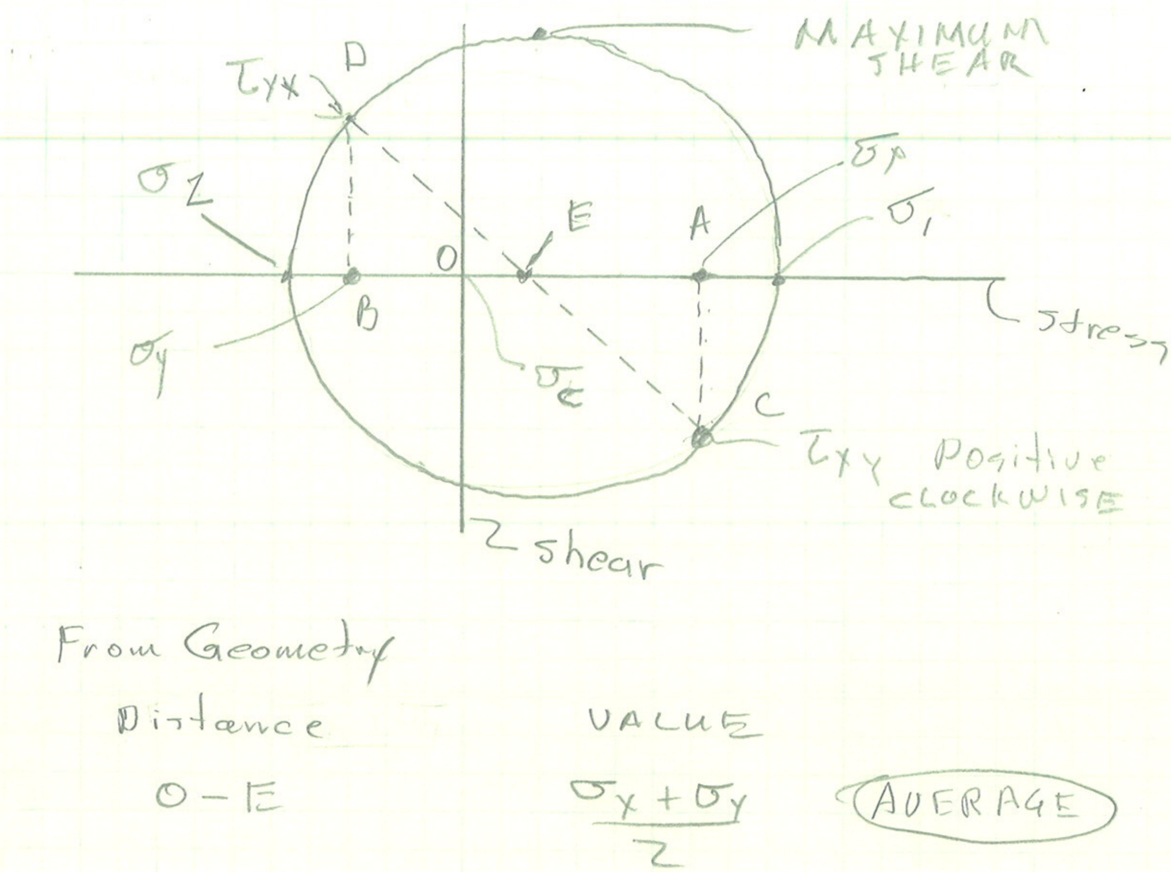
Tensile normal stresses are positive and compressive stresses are negative.

We can generalize the sign of both normal and shear stresses

The sign is positive if the signs of the surface normal and its stress direction are the same, and negative if they are different.

FOR 2D - PLANE STRESS





From Geometry

Distance	VALUE	
O - E	$\dfrac{\sigma_x + \sigma_y}{2}$	AVERAGE
E - A	$\dfrac{\sigma_x - \sigma_y}{2}$	TRIANGLE BASE

E-C $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

We usually arrange the stresses so that

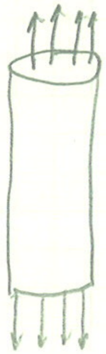
$$\sigma_1 > \sigma_2$$

MAXIMUM SHEAR

The maximum shear is:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

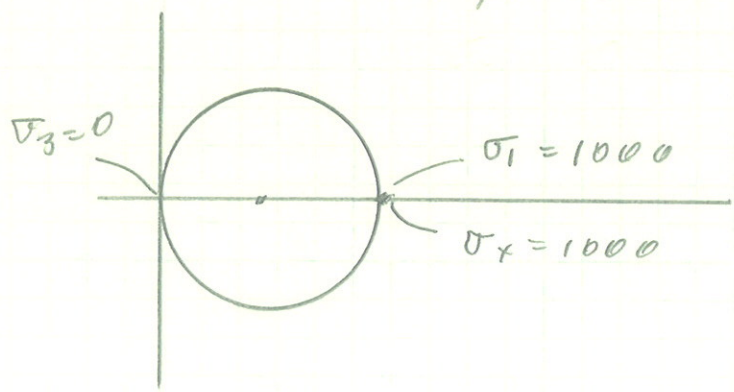
ONE DIMENSIONAL EXAMPLE



We have test sample that is cylinderical in shape. We are applying an axial stress of 1000 MPa

$\sigma_x = 1000\ MPa$

$\sigma_y = 0 \qquad \tau_{xy} = 0$



Using the formulas we developed

$$\sigma_1 = \frac{\sigma_x + \sigma_x}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = \frac{1000 + 0}{2} + \sqrt{\left(\frac{1000 - 0}{2}\right)^2 + 0} = 500 + 500$$

$\underline{\sigma_1 = 1000\ MPa}$

$$\sigma_3 = \frac{\sigma_x - \sigma_y}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_3 = \frac{1000 - 0}{2} - \sqrt{\left(\frac{1000}{2}\right)^2 + 0} = 500 - 500$$

$\underline{\sigma_3 = 0}$

MAXIMUM SHEAR

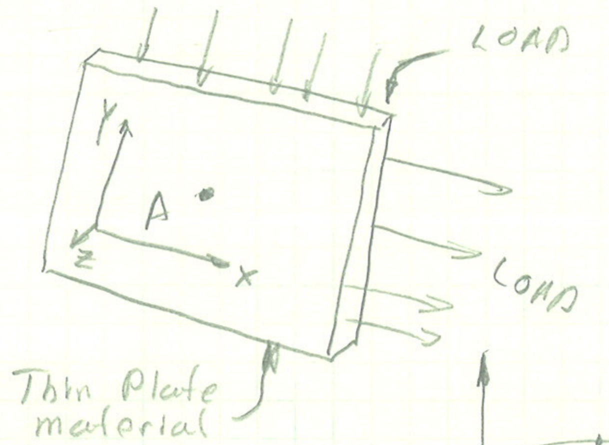
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{1000 - 0}{2}\right)^2 + 0^2} =$$

$$\tau_{max} = 500 \text{ MPa}$$

The applied stress is purely normal tension but this produces a significant shear stress.

2 DIMENSIONAL EXAMPLE

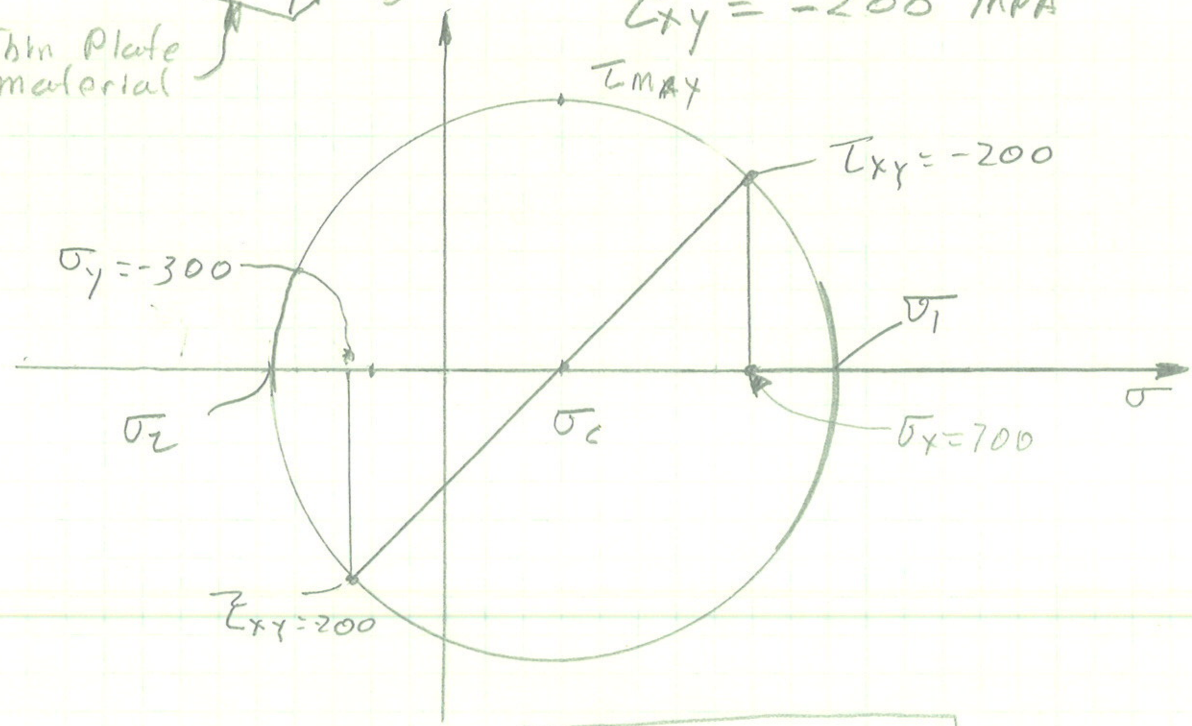


The stress at A is

$\sigma_x = 700$ MPA

$\sigma_y = -300$ MPA

$\tau_{xy} = -200$ MPA



$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = \frac{700 - 300}{2} + \sqrt{\left(\frac{700 + 300}{2}\right)^2 + 200^2}$$

$$\sigma_1 = 200 + \sqrt{500^2 + 200^2}$$

$$\boxed{\sigma_1 = 738.5 \text{ MPA}}$$

TOPS 35500

$$\sigma_3 = \frac{\sigma_Y + \sigma_Y}{2} - \sqrt{\left(\frac{\sigma_Y - \sigma_Y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 200 - \sqrt{500^2 + 200^2}$$

$$\boxed{\sigma_3 = -338.5 \text{ MPA}}$$

MAXIMUM SHEAR

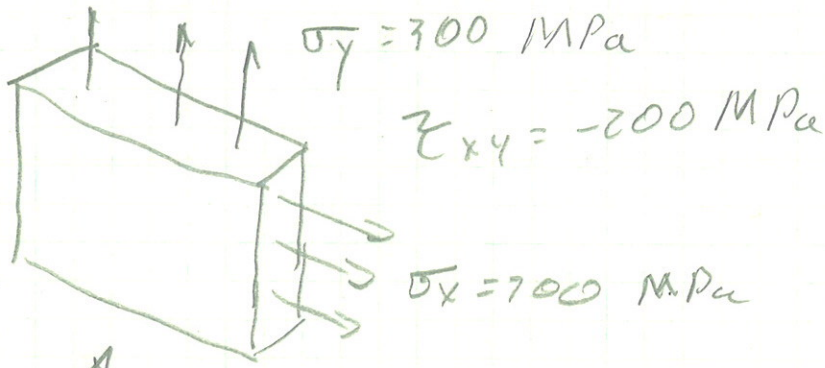
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{700 + 300}{2}\right)^2 + 200^2}$$

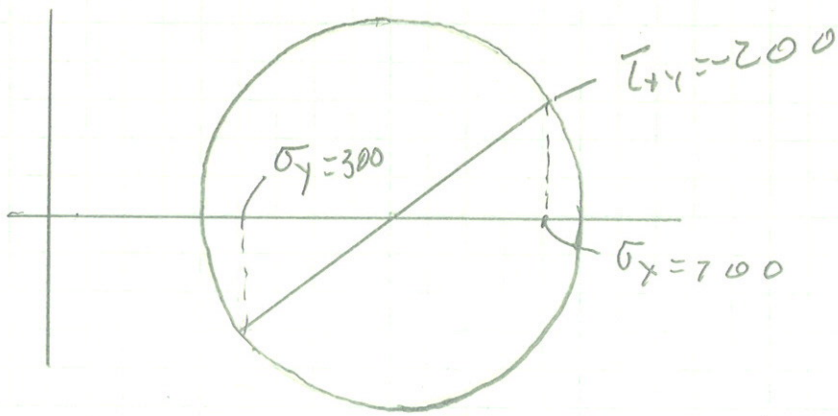
$$= \sqrt{500^2 + 200^2}$$

$$\boxed{\tau_{max} = 538.5 \text{ MPA}}$$

LOOKING AT A PREVIOUS EXAMPLE



Both in Tension



$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = \frac{700 + 300}{2} + \sqrt{\left(\frac{700 - 300}{2}\right)^2 + (-200)^2}$$

$$\sigma_1 = 500 + 282.8 = \underline{782.8 \text{ MPa}}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = 500 - 282.8 = 217.2 \text{ MPa}$$

Now what is the maximum shear?

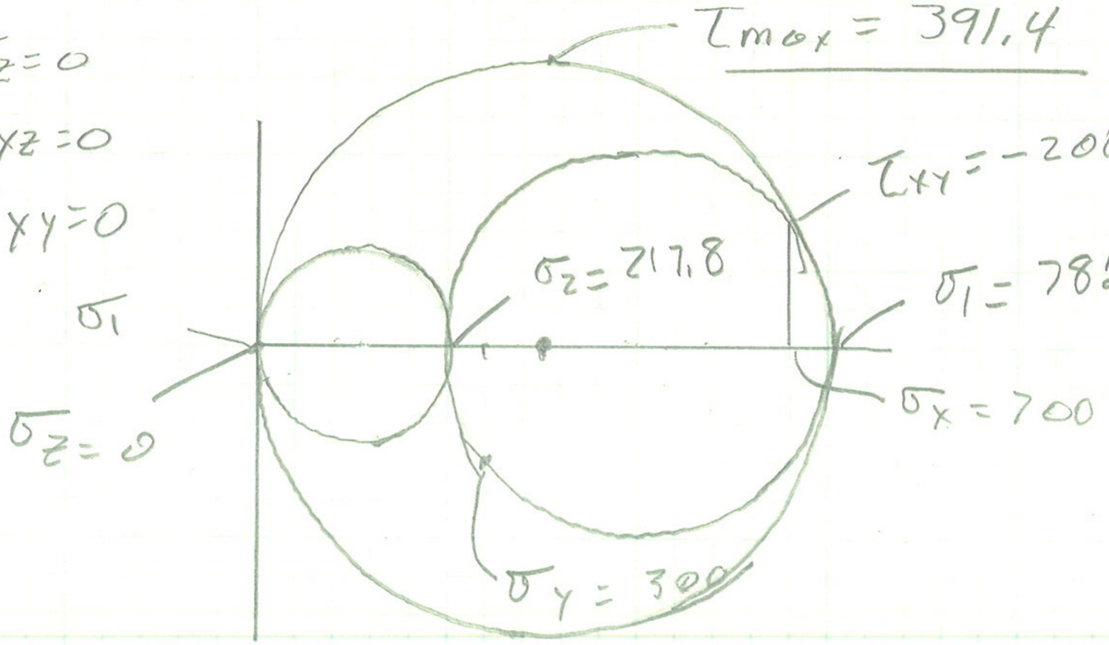
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

But, we live in a 3D world - how about the stresses in the z direction

$\sigma_z = 0$

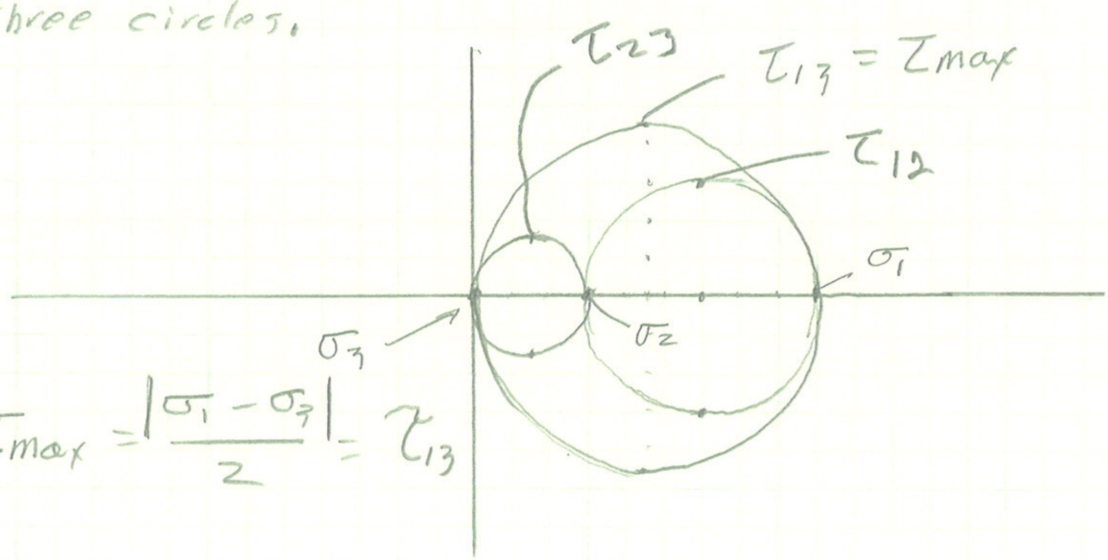
$\tau_{yz} = 0$

$\tau_{xy} = 0$



$$\tau_{max} = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{782.8 - 0}{2} = 391.4$$

In this case minimum and maximum principal stresses and the maximum shear stress did not change. If σ_y is not negative as it was in the previous case one must create all three circles.



$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \tau_{13}$$

3D PRINCIPAL STRESSES

Mohr's circle can not be used for the general 3D case. Here we use the formulas derrived from tensor algebra

Principal stresses $\rightarrow$ $\sigma^3 - C_2\sigma^2 - C_1\sigma - C_0 = 0$

Where

$$C_2 = \sigma_x + \sigma_y + \sigma_z$$

$$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$$

$$C_0 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zy} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zy}^2 - \sigma_z\tau_{xy}^2$$

This cubic equation can be solved with an advanced calculator or with MATLAB.

The principal shear stresses can be computed with

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$

$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2}$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

EXAMPLE OF A 3D STRESS

A 3D element has the following stresses.

NORMAL STRESS		SHEAR STRESS	
σ_x	-500	τ_{xy}	100
σ_y	750	τ_{yz}	250
σ_z	250	τ_{zx}	1000

$$\sigma^3 - C_2\sigma^2 - C_1\sigma - C_0 = 0$$

$$C_2 = \sigma_x + \sigma_y + \sigma_z = -500 + 750 + 250 = 500$$

$$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$$

$$= 100^2 + 250^2 + 1000^2 + 500\cdot750 - 750\cdot250 + 500\cdot250$$

$$= 1\,385\,000$$

$$C_0 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$C_0 = -500\cdot750\cdot250 + 2\cdot100\cdot250\cdot1000 + 500\cdot250^2 - 750\cdot1000^2 - 250\cdot100^2$$

$$C_0 = -765\,000\,000$$

We can solve the cubic polynomial using either a calculator or Matlab. Matlab is easier so

$$C = [1 \; -500 \; -1{,}385{,}000 \quad 765{,}000{,}000]$$

$$r = \text{roots}(c)$$

$r_1 = 568.3$ $\quad$ $\sigma_1 = 1126.6$

$r_2 = 1126.6$ $\quad$ $\sigma_2 = 568.3$

$r_3 = -1194.9$ $\quad$ $\sigma_3 = -1194.9$

Note that $\sigma_1 \geq \sigma_2 \geq \sigma_3$ so we must order them in this order.

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2} = 1160.8$$ SHEAR MAX

$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} = -279.2$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2} = -881.6$$

COMPARISON BETWEEN 2D AND 3D

For a 2D case the normal and shear stresses are zero along one of the axes. We will assume they are zero along the z axis

DRAW PICTURE

$\sigma_z = 0 \qquad \tau_{zx} = 0 \qquad \tau_{yz} = 0$ $\quad$ PLANE STRESS

our equations become

$$C_2 = \sigma_x + \sigma_y + \cancel{\sigma_z} = \sigma_x + \sigma_y$$

$$C_1 = \tau_{xy}^2 + \cancel{\tau_{yz}^2}^{0} + \cancel{\tau_{zx}^2}^{0} - \sigma_x\sigma_y - \cancel{\sigma_y\sigma_z}^{0} - \cancel{\sigma_z\sigma_x}^{0}$$

$$C_1 = \tau_{xy}^2 - \sigma_x\sigma_y$$

$$C_0 = \cancel{\sigma_x\sigma_y\sigma_z}^{0} + \cancel{2\tau_{xy}\tau_{yz}\tau_{zx}}^{0} - \cancel{\sigma_x\tau_{yz}^2}^{0} - \cancel{\sigma_y\tau_{zx}^2}^{0} - \cancel{\sigma_z\tau_{xy}^2}^{0}$$

so

$$C_0 = 0$$

C_0 is zero so our polynomial becomes

$$\sigma^3 - C_2\sigma^2 - C_1\sigma = 0$$

dividing through by σ yields

$$\sigma^2 - C_2\sigma - C_1 = 0$$

we can solve this using the quadratic formula

$$\sigma_{1,2} = \frac{C_2 \pm \sqrt{C_2^2 + 4C_1}}{2}$$

$$\sigma_{1,2} = \frac{C_2}{2} \pm \sqrt{\left(\frac{C_2}{2}\right)^2 + C_1}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2 - \sigma_x\sigma_y}$$

combining terms

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{\sigma_x^2 - 2\sigma_x\sigma_y + \sigma_y^2}{4} + \tau_{xy}^2}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2}$$

$$\boxed{\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

same as the 2D problem
Derived from MOHR's Circle

HOMEWORK

4-1

a C I J

Due Wednesday Feb 23
@ 16h00

4.1a $\sigma_x = 1000 \quad \sigma_y = 0 \quad \sigma_z = 0 \quad \tau_{xy} = 500$

$\tau_{yz} = 0 \quad \tau_{zx} = 0$

2D problem

$$\sigma^3 - C_2\sigma^2 - C_1\sigma - C_0 = 0$$

$$C_2 = \sigma_y + \sigma_y + \sigma_z = 1000$$

$$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$$

$$C_1 = 500^2 = 250000$$

$$C_0 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$C_0 = 0$$

Our equation becomes

$$\sigma^3 - 1000\sigma^2 - 250000\sigma - 0 = 0$$

Divide through by σ

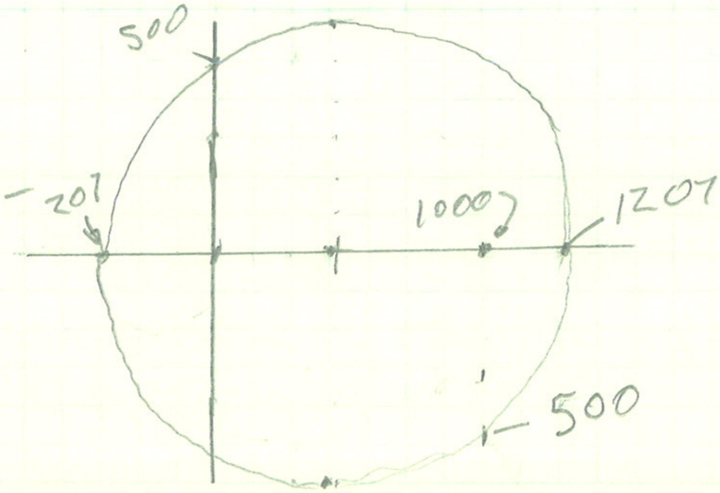
$$\sigma^2 - 1000\sigma - 250000 = 0$$

$r_1 = -207 \qquad r_2 = 1207$

so $\sigma_1 = 1207 \quad \sigma_2 = 0 \quad \sigma_3 = -207$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{1207 + 207}{2}$$

$\tau_{max} = 707$



4.1 C $\sigma_x = 500 \quad \sigma_y = -500 \quad \tau_{xy} = 1000$

$$\sigma^3 - C_2\sigma^2 - C_1\sigma - C_0 = 0$$

$$C_2 = \sigma_x + \sigma_y + \sigma_z = 0$$

$$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_x\sigma_z - \sigma_z\sigma_x$$

$$= 1000^2 - (500)(-500) = 1{,}250{,}000$$

$$C_0 = \sigma_x \cancel{\sigma_y} \sigma_z + 2\tau_{xy}\cancel{\tau_{yz}}\tau_{zx} - \cancel{\sigma_x}\tau_{yz}^2 - \cancel{\sigma_y}\tau_{zx}^2 - \cancel{\sigma_z}\tau_{xy}^2$$

$$C_0 = 0$$

Our equation becomes

$$\sigma^3 - 1{,}250{,}000\sigma = 0$$

or

$$\sigma^2 - 1{,}250{,}000 = 0$$

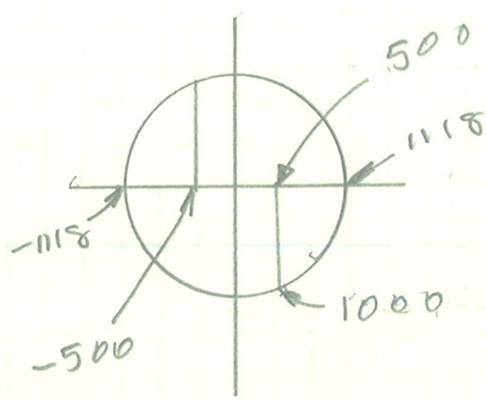
$$\sigma^2 = 1{,}250{,}000$$

$$\sigma = \pm 1118$$

so

$$\sigma_1 = 1118 \quad \sigma_3 = -1118 \quad \sigma_2 = 0$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{1118 + 1118}{2} = \underline{1118}$$



4.1.i $\sigma_x = 1000 \quad \sigma_y = -250 \quad \sigma_z = -750$

$\tau_{xy} = 250 \quad \tau_{yz} = 500 \quad \tau_{zx} = 750$

$\sigma^3 - C_2\sigma^2 - C_1\sigma - C_0 = 0$

$C_2 = \sigma_x + \sigma_y + \sigma_z = 1000 - 250 - 750 = \underline{0}$

$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$

$C_1 = 250^2 + 500^2 + 750^2 + 1000 \cdot 250 - 250 \cdot 750 + 1000 \cdot 750$

$\underline{C_1 = 1687500}$

$C_0 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 -$
$\sigma_z\tau_{xy}^2$

$C_0 = 1000 \cdot (-250) \cdot (-750) + 2 \cdot 250 \cdot 500 \cdot 750 - 1000 \cdot 500^2 +$
$250 \cdot 750^2 - 750 \cdot 250^2$

$\underline{C_0 = 218750000}$

Our Equation is

$\sigma^3 - 1{,}687{,}500\,\sigma - 218{,}750{,}000$

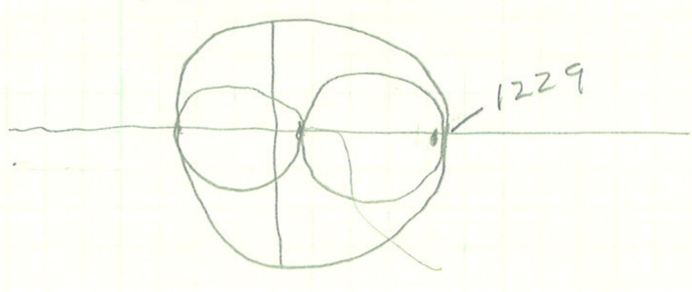
$r_1 = 131 \quad r_2 = 1229 \quad r_3 = -1360$

so

$\underline{\sigma_1 = 1229} \quad \underline{\sigma_2 = 131} \quad \underline{\sigma_3 = -1360}$

$\tau_{max} = \dfrac{|\sigma_1 - \sigma_3|}{2} = \dfrac{1229 + 1360}{2}$

$\underline{\tau_{max} = 744.5}$



4.1.j $\sigma_x = -500 \quad \sigma_y = 750 \quad \sigma_z = 250$

$\tau_{xy} = 100 \quad \tau_{yz} = 250 \quad \tau_{zx} = 1000$

$\sigma^3 - C_2\sigma^2 - C_1\sigma - C_0 = 0$

$C_2 = \sigma_x + \sigma_y + \sigma_z$

$C_2 = -500 + 750 + 250 = \underline{500}$

$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$

$= 100^2 + 250^2 + 1000^2 + 500(750) - 750(250) + 250(500)$

$C_1 = 1385000$

$C_0 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$

$= (-500)(750)(250) + 2\cdot 100\cdot 250\cdot 1000 + 500(250)^2 - 750(1000)^2 - 250(100)^2$

$C_0 = 765{,}000{,}000$

$C = [1, -500, -1385000, -765\,000\,000]$

$R_1 = 568 \quad R_2 = 1126 \quad R3 = -1194$

so

$\underline{\sigma_1 = 1126 \quad \sigma_2 = 568 \quad \sigma_3 = -1194}$

$\tau_{max} = \dfrac{\sigma_1 - \sigma_3}{2} = \dfrac{1126 + 1194}{2}$

$\underline{\tau_{max} = 1160}$