Type-3: $F(z, p, q) = 0$ [x, y not known explicitly]

(*) Let $z = f(x+ay)$ be the soln.

(*) put $x+ay = u \Rightarrow z = f(u)$

(*) Diff partially w.r.t. x and y.
Find p and q

(*) Sub in ① and solve ODE using Variable Separable Method. $\Rightarrow$ C.I of $F(z, p, q) = 0$

(*) G.I & S.I is as usual.

1) Solve $p(1+q) = qz$. —— ①

Given: ① is of the form $F(p, q, z) = 0$.

Let $z = f(x+ay)$ be the soln of ①.

~~Let~~ put $x+ay = u$

$$\frac{\partial u}{\partial x} = 1 \text{ and } \frac{\partial u}{\partial y} = a \quad \text{—— ②}$$

Now $z = f(u)$

$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot 1 \\ q &= \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} \cdot a \end{aligned}\right\} \text{—— ③}$$

Sub ③ in ①,

$$\frac{dz}{du}\left(1 + \frac{dz}{du} \cdot a\right) = a \cdot \frac{dz}{du} \cdot z.$$

$$1 + a\frac{dz}{du} = a\,\cancel{\frac{dz}{du}}\, z.$$

$$\frac{dz}{du} = \frac{az - 1}{a}$$

$$\frac{dz}{du} = z - \frac{1}{a}.$$

$$\frac{dz}{z - \frac{1}{a}} = du.$$

Variable Separable Method

$$\log\left(z - \frac{1}{a}\right) = u + c \quad \text{—— ④}$$

$$\log\left(z - \frac{1}{a}\right) = x + ay + c$$

which is the C.I.

G.I put $c = f(a)$ in ④, we get

$$\log\left(z - \frac{1}{a}\right) = x + ay + f(a) \quad \text{—— (5)}$$

$$\left(\frac{1}{z - \frac{1}{a}}\right)\left(\frac{\partial z}{\partial a} + \frac{1}{a^2}\right) = 0 + y + f'(a)$$

$$f'(a) + y = 0 \quad \text{—— (6)}$$

Eliminating 'a' from ⑤ & ⑥ gives G.I.

S.I Diff ④ partially w.r. to 'a'

$$\frac{1}{(z - \frac{1}{a})}\left(\frac{\partial z}{\partial a} + \frac{1}{a^2}\right) = 0 + y + 0$$

$$\Rightarrow y = \frac{1}{(z - \frac{1}{a})}\left(\frac{\partial z}{\partial a} + \frac{1}{a^2}\right) \quad \text{—— (7)}$$

Eliminating 'a' from ④ & ⑦ gives S.I.

HW:

1) $z^2 = p^2 + q^2 + 1$

2) $ap + bq + cz = 0$.

3) find C.I & S.I of $p^3 + q^3 = 8z$.

4) $9pqz^4 = 4(1 + z^3)$.

5) $q^2 = z^2 p^2 (1 - p^2)$

6) $p^2 + pq = z^2$.