Problems on Vibrating string with non-zero Intial Velocity.

Problem 1:

A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$, then show that $y(x,t)=\frac{8\lambda l^3}{\pi^4 a}\sum_{n=1,3,5}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin$

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

Step 1:

The boundary conditions are

(a) $y(0,t)=0$ for all $t>0$

(b) $y(l,t)=0$ for all $t>0$

(c) $y(x,0)=0$ for all x in $(0,l)$

[$\because$ The string when $t=0$ is in its equilibrium position and hence there is no displacement]

(d) $\frac{\partial y(x,0)}{\partial t}=\lambda x(l-x)$ for every x in $(0,l)$.

Step 2: Choose the solution

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat) \quad (1)$$

Step 3: Applying condition (a) in (1).

$y(0,t)=0$ put $x=0$; $t=t$.

$$y(0,t)=(c_1\cos 0+c_2\sin 0)(c_3\cos pat+c_4\sin pat)$$

$$\Rightarrow c_1(\cos pat+c_4\sin pat)=0$$

$$c_1=0 \text{ and } \cos pat+c_4\sin pat\neq 0$$

Sub $c_1 = 0$ in ①

$$y(x,t) = c_2 \sin px \,(c_3 \cos pat + c_4 \sin pat) \quad \text{——— ②}$$

Step 4: Applying (b) in ②

$y(l,t) = 0 \qquad x = l \,;\, t = t$

$y(l,t) = c_2 \sin pl \,(c_3 \cos pat + c_4 \sin pat) = 0$

$\Rightarrow c_2 \sin pl = 0$ and $c_3 \cos pat + c_4 \sin pat \neq 0$.

$c_2 = 0$ (or) $\sin pl = 0$.

If $c_2 = 0$, the solution is trivial.

$\therefore \sin pl = 0$

$\Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$

Sub $p = \frac{n\pi}{l}$ in ②,

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi}{l}at + c_4 \sin \frac{n\pi}{l}at\right) \quad \text{——— ③}$$

Step 5: Applying (c) in ③.

$y(x,0) = 0 \Rightarrow x = x \,;\, t = 0$

$c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos(0) + c_4 \sin(0)\right) = 0$

$c_2 c_3 \sin \frac{n\pi x}{l} = 0$.

$c_2 \neq 0$ and $c_3 \sin \frac{n\pi x}{l} = 0$

Here $\sin \frac{n\pi x}{l} \neq 0$ (since it is defined for all x)

$\therefore \boxed{c_3 = 0}$

Sub $c_3 = 0$ in ③.

$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(0 + c_4 \sin \frac{n\pi at}{l}\right)$

$y(x,t) = c_2 c_4 \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$

$y(x,t) = c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$ ——— ④

Step 6: Applying condition (d) in (4).

$y(x,t)$

$$\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}\right) = \lambda x(l-x)$$

$$\sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} = \lambda x(l-x)$$

put $x=x$ and $t=0$.

$$\sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l}\cos(0) = \lambda x(l-x)$$

$$\sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l} = \lambda x(l-x) \quad \text{--- (5)}$$

To find c_n, expand $\lambda x(l-x)$ in a half-range Fourier sine series we get,

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--- (6)}.$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$.

On Comparing the eqns (5) & (6)

$$c_n\left(\frac{n\pi a}{l}\right) = b_n \Rightarrow c_n = b_n\left(\frac{l}{n\pi a}\right) \quad \text{(7)}$$

$$C_n = b_n\left(\frac{1}{n\pi a}\right) \quad \text{——— (6)}$$

$$b_n = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx-x^2)\left(\frac{-l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right) - (l-2x)\left(\frac{-l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right) - 2\left(\frac{l^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{l}\right)\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left(0-0-\frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(0+0-\frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2\lambda}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$b_n = \frac{4\lambda l^3}{l\,n^3\pi^3}\left[-(-1)^n+1\right] \qquad \therefore\ b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore\ b_n = \begin{cases} \dfrac{8\lambda l^2}{n^3\pi^3}, & \text{when } n \text{ is odd,} \\ 0, & \text{when } n \text{ is even.} \end{cases}$$

$\therefore$ sub b_n in (6),

$$C_n = \frac{8\lambda l^2}{n^3\pi^3}\,\frac{1}{n\pi a}$$

$$C_n = \frac{8\lambda l^2}{n^4\pi^4 a} \quad \text{——— (7)}$$

Sub C_n in (4),

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8\lambda l^2}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$

$$\therefore\ y(x,t) = \frac{8\lambda l^2}{\pi^4 a}\sum_{n=1,3,5}^{\infty}\left(\frac{1}{n^4}\right)\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l} \quad \text{——— (8)}$$

2) If a string of length 'l' is initially at rest in its equilibrium position and each of its points is given the velocity

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}, \quad 0 < x < l.$$

Determine the displacement function $y(x,t)$.

Solution: The Wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Step 1: The boundary conditions are

a) $y(0,t) = 0$ forevery $t > 0$

b) $y(l,t) = 0$ for every $t > 0$

c) $y(x,0) = 0$ for all x in $(0,l)$

d) $\frac{\partial y(x,0)}{\partial t} = v_0 \sin^3 \frac{\pi x}{l}$ for all x in $(0,l)$

Same as problem 1 ~~as~~ till step 5.

Step 6: $y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l}$ —— (4)

Applying condition (d) in (4)

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} (1) = v_0 \sin^3 \frac{\pi x}{l}$$

$$\Rightarrow \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = v_0 \sin^3 \frac{\pi x}{l} \quad \text{—— (5)}$$

$$C_1 \frac{\pi a}{l} \sin \frac{\pi x}{l} + C_2 \frac{2\pi a}{l} \sin \frac{2\pi x}{l} + C_3 \frac{3\pi a}{l} \sin \frac{3\pi x}{l}$$

$$= v_0 \left[\frac{1}{4} \left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right)\right]$$

$$= \frac{3v_0}{4} \sin \frac{\pi x}{l} - \frac{v_0}{4} \sin \frac{3\pi x}{l}$$

On Equating like coe. we get,

$$c_1 \frac{\pi a}{l} = \frac{3v_0}{4} \quad \Big| \quad c_2 = 0 \quad \Big| \quad c_3 \frac{3\pi a}{l} = -\frac{v_0}{4}$$

$$c_1 = \frac{3v_0 l}{4\pi a} \quad \Big| \quad c_3 = -\frac{l v_0}{12\pi a}$$

Sub c_1 and c_2

$$y(x,t) = \frac{3v_0 l}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a l}{l}$$

is the solution.

3) A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities v where

$$v = \begin{cases} \frac{cx}{l}, & \text{in } 0 < x < l \\ \frac{c}{l}(2l - x), & \text{in } l < x < 2l \end{cases}$$

x being the distance from an end point. Find the displacement of the string at any time.

Solution:

The wave eqn is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

Let $2l = L$ for convenience.

Step 1: The boundary conditions are

a) $y(0,t) = 0 \ \forall t > 0$

b) $y(L,t) = 0 \ \forall t > 0$

c) $y(x,0) = 0 \ \forall x$ in $(0,L)$

d) $$\frac{\partial y}{\partial t}(x,0) = \begin{cases} \frac{2cx}{L}, & 0 < x < \frac{L}{2} \\ \frac{2c}{L}(L - x), & \frac{L}{2} < x < L \end{cases}$$

Step 2: choose the solution

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \text{(1)}$$

Step 3: (a) in (1)

$\Rightarrow c_1 = 0$ $\quad y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat)$

Step 4: Applying (b) condition (2)

$$y(L,t)=0 \qquad x=L \;/\; t=t$$

$$y(L,t) = c_2 \sin pL \left[c_3 \sin pat + c_4 \sin pat \right] = 0$$

$$\Rightarrow \boxed{p = \frac{n\pi}{L}}$$

put $p = \frac{n\pi}{L}$ in (2).

$$y(x,t) = c_2 \sin \frac{n\pi x}{L} \left(c_3 \cos \frac{n\pi at}{L} + c_4 \sin \frac{n\pi at}{L} \right) \quad \text{(3)}$$

Step 5: Applying (c) in (3)

$$y(x,0)=0 \qquad \Rightarrow c_3 = 0.$$

$$y(x,t) = c_2 \sin \frac{n\pi x}{L} \, c_4 \sin \frac{n\pi at}{L}$$

$$= c_2 c_4 \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L}$$

$$\therefore y(x,t) = c_n \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \quad \text{——— (4)}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \quad \text{——— (5)}$$

Step 6: Applying (d) in (5)

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{L} \right) \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}$$

$$\frac{\partial y}{\partial t}(x,t) = \quad '' \quad = f(x)$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{L} \right) \sin \frac{n\pi x}{L} \cancel{\cos \frac{n\pi at}{L}} = f(x) = \begin{cases} \frac{2cx}{L} \\ \frac{2c}{L}(L-x) \end{cases} \quad \text{(6)}$$

To find c_n, expand $f(x)$ in a half range fourier Sine Series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad \text{——— (7)}$$

Step 4: Applying (b) condition (2)

$y(L,t)=0 \qquad x=L \;|\; t=t$

$y(L,t) = C_2 \sin pL \left[C_3 \sin pat + C_4 \sin pat\right] = 0$

$\Rightarrow \boxed{p = \frac{n\pi}{L}}$

put $p = \frac{n\pi}{L}$ in (2).

$y(x,t) = C_2 \sin\frac{n\pi x}{L}\left(C_3 \cos\frac{n\pi at}{L} + C_4 \sin\frac{n\pi at}{L}\right)$ — (3)

Step 5: Applying (c) in (3)

$y(x,0)=0 \quad \Rightarrow C_3 = 0.$

$y(x,t) = C_2 \sin\frac{n\pi x}{L}\, C_4 \sin\frac{n\pi at}{L}$

$= C_2 C_4 \sin\frac{n\pi x}{L} \sin\frac{n\pi at}{L}$

$\therefore y(x,t) = c_n \sin\frac{n\pi x}{L} \sin\frac{n\pi at}{L}$ — (4)

The most general solution is

$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \sin\frac{n\pi at}{L}$ — (5)

Step 6: Applying (d) in (5)

$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{L}\right) \sin\frac{n\pi x}{L} \cos\frac{n\pi at}{L}$

$\frac{\partial y}{\partial t}(x,t) = \quad '' \quad = f(x)$

$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{L}\right) \sin\frac{n\pi x}{L} \cancel{\cos\frac{n\pi at}{L}} = f(x) = \begin{cases} \frac{2cx}{L} \\ \frac{2c}{L}(L-x) \end{cases}$ — (6)

To find c_n, expand $f(x)$ in a half range Fourier Sine Series,

$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ — (7)

From (6) & (7) we get

$c_n \frac{n\pi a}{L} = b_n \Rightarrow$

Now, $b_n = \frac{2}{L}\int_0^L f(x) \sin$

$= \frac{2}{L}\left[\int_0^{L/2} f(x) \sin\frac{n\pi x}{L}\right.$

$= \frac{2}{L}\left[\int_0^{L/2} \frac{2cx}{L} \sin\frac{n\pi x}{L}\right.$

$= \frac{4c}{L^2}\left[\int_0^{L/2} x \sin\frac{n\pi x}{L}\, dx\right.$

$u = x$	$u = L-x$	$dv = $
$u' = 1$	$u' = -1$	$v = $
$u'' = 0$	$u'' = 0$	$v_1 = $

$= \frac{4c}{L^2}\left[\left(x\left(\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right)\right.\right.$

$+ \left((L-x)\left(\frac{-L}{n\pi}\cos\right.\right.$

$= \frac{4c}{L^2}\left[\left(\cancel{\frac{L}{2}\left(\frac{-L}{n\pi}\cos\frac{n\pi}{2}\right)}\right.\right.$

$+ \left(0+0-\cancel{\left(\frac{L}{2}\left(\frac{-L}{n\pi}\right.\right.}\right.$

$= \frac{4C}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\right.$

$= \frac{8C}{L^2}\,\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}.$

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from ⑥ & ⑦ we get

$$C_n \frac{n\pi a}{L} = b_n \Rightarrow C_n = \frac{b_n L}{n\pi a} \quad \text{—— (8)}$$

Now, $b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$

$$= \frac{2}{L}\left[\int_0^{L/2} f(x) \sin \frac{n\pi x}{L}\, dx + \int_{L/2}^{L} f(x) \sin \frac{n\pi x}{L}\, dx\right]$$

$$= \frac{2}{L}\left[\int_0^{L/2} \frac{2Cx}{L} \sin \frac{n\pi x}{L}\, dx + \int_{L/2}^{L} \frac{2C}{L}(L-x) \sin \frac{n\pi x}{L}\, dx\right]$$

$$= \frac{4C}{L^2}\left[\int_0^{L/2} x \sin \frac{n\pi x}{L}\, dx + \int_{L/2}^{L} (L-x) \sin \frac{n\pi x}{L}\, dx\right]$$

$u = x$ | $u = L - x$ | $dv = \sin \frac{n\pi x}{L}\, dx$

$u' = 1$ | $u' = -1$ | $v = \frac{-L}{n\pi} \cos \frac{n\pi x}{L}$

$u'' = 0$ | $u'' = 0$ | $v_1 = \frac{-L^2}{n^2\pi^2} \sin \frac{n\pi x}{L}$

$$= \frac{4C}{L^2}\left[\left(x\left(\frac{-L}{n\pi}\cos \frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi x}{L}\right)_0^{L/2} + \left((L-x)\left(\frac{-L}{n\pi}\cos \frac{n\pi x}{L}\right) + \frac{-L^2}{n^2\pi^2} \sin \frac{n\pi x}{L}\right)_{L/2}^{L}\right]$$

$$= \frac{4C}{L^2}\left[\left(\cancel{\frac{L}{2}\left(\frac{-L}{n\pi}\cos \frac{n\pi}{2}\right)} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right) - (0) + \left(0 + 0 - \left(\cancel{\frac{L}{2}\left(\frac{-L}{n\pi}\cos \frac{n\pi}{2}\right)} + \frac{-L^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{4C}{L^2}\left[\frac{2L^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right]$$

$$= \frac{8C}{L^2} \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

$$\therefore b_n = \frac{8C}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

Sub b_n in ⑧,

$$C_n = \frac{\frac{8C}{n^2\pi^2} \sin \frac{n\pi}{2} L}{n\pi a} = \frac{8CL}{n^3\pi^3 a} \sin \frac{n\pi}{2}$$

Replace L by $2l$ $\quad C_n = \frac{16cl}{n^3\pi^3 a}\sin\frac{n\pi}{2}$ ——— (9)

Sub (9) in (5),

$$y(x,t) = \sum_{n=1}^{\infty}\frac{16cl}{n^3\pi^3 a}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}$$

$$y(x,t) = \frac{16cl}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$$

is the solution.