Unit - V

Z - Transform

Definition: (Two sided or Bilateral Z-transform)

Let $\{f(n)\}$ be a sequence defined for all integers ($n = 0, \pm 1, \pm 2, \ldots$) then Z-transform is defined as

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n} = F(z) \quad (z \to \text{a complex number})$$

Definition: (One sided Z-transform)

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z) \quad (z \to \text{a complex number})$$

Definition: (Z-transform for discrete values of t

If $f(t)$ is defined for discrete values of 't' where $t = nT$, $n = 0, 1, 2, 3, \ldots$ T being the sampling period, then

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} = F(z)$$

Note: 1) If $f(n)$ is gn then keep 'n' as it is.

2) If $f(t)$ is gn then Replace 't' by 'nT'.

3) $F(z)$ and $f(n)$ are called Z-transform pair.

Formulae:

i) If $Z[f(n)] = F(z)$ then $Z^{-1}[F(z)] = f(n)$

1) $Z[a^n] = \frac{z}{z-a}$

2) $Z[(-a^n)] = \frac{z}{z+a}$

3) $Z[n] = \frac{z}{(z-1)^2}$

4) $Z[na^n] = \frac{az}{(z-a)^2}$

5) $Z[na^{n-1}] = \frac{z}{(z-a)^2}$

6) $Z[a^{n-1}] = \frac{1}{z-a}$

7) $Z[(-a)^{n-1}] = \frac{1}{z+a}$

8) $Z[n(-a)^{n-1}] = \frac{z}{(z+a)^2}$

9) $Z[(n-1)a^{n-2}] = \frac{1}{(z-a)^2}$

10) $Z[(n-1)(-a)^{n-2}] = \frac{1}{(z+a)^2}$

11) $Z\left[\frac{1}{n+1}\right] = -z\log\left(1-\frac{1}{z}\right)$

12) $Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$

13) $Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

14) $Z\left[\frac{1}{n!}\right] = e^{1/z}$

15) $Z[n^2] = \frac{z(z+1)}{(z-1)^3}$

Find Z-transform: Problems

1) 1 2) n 3) $\frac{1}{n}$ 4) $\frac{1}{n+1}$ 5) $\frac{1}{n+2}$ 6) $\frac{1}{n-1}$

7) $\frac{1}{n!}$ 8) $\frac{1}{(n+1)!}$ (9) a^n

Soln: 1) $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z[1] = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1-\frac{1}{z}\right)^{-1} \text{ for } \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{\left(1-\frac{1}{z}\right)}$$

formula: $(1-x)^{-1} = 1+x+x^2+\cdots$ if $|x|<1$

Here $x = \frac{1}{z}$

$|x|<1 \Rightarrow \left|\frac{1}{z}\right| < 1 \Rightarrow 1 < |z|$

$$z[1] = \frac{1}{\left(\frac{z-1}{z}\right)} = \frac{z}{z-1}$$

$$z[1] = \frac{z}{z-1}$$

2) $$z[n] = \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right]$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-2}$$

formula: $(1-x)^{-2} = 1+2x+3x^2+\cdots$ if $|x|<1$

Here $x = \frac{1}{z}$ ie $\left|\frac{1}{z}\right| < 1 \Rightarrow |z|>1$

$$z[n] = \frac{1}{z}\left(\frac{1}{\left(1-\frac{1}{z}\right)^2}\right) = \frac{1}{z}\left(\frac{z}{z-1}\right)^2$$

$$= \frac{z^2}{z(z-1)^2} = \frac{z}{(z-1)^2}$$

$$\therefore z[n] = \frac{z}{(z-1)^2}$$

3) $z\left(\frac{1}{n}\right)$, $|z|>1$, $n>0$

$$z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \quad (\because n>0)$$ (we put $n=1$ to ∞)

$$= z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \cdots$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$$

formula: $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ if $|x|<1$

$= -\log\left(1-\frac{1}{z}\right)$ formula: Here $x = \frac{1}{z}$, $\left|\frac{1}{z}\right| < 1$

$|z| > 1$.

$= \log\left(1-\frac{1}{z}\right)^{-1}$

$Z\left(\frac{1}{n}\right) = \log\left(\frac{z-1}{z}\right)^{-1} = \log\left(\frac{z}{z-1}\right)$

$\therefore Z\left[\frac{1}{n}\right] = \log\left[\frac{z}{z-1}\right]$

formula for $\frac{1}{n+1}$ $\quad -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

formula for $\frac{1}{n+2}$ $\quad n\log m = \log m^n$

$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$ | $Z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$

$Z\left[\frac{1}{n-1}\right] = \frac{1}{z}\log\left[\frac{z}{z-1}\right]$.

Example 7: Find $Z\left[\frac{1}{n!}\right]$

Soln: $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{1!}z^{-1} + \frac{1}{2!}z^{-2} + \cdots$

$= 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \cdots$

$\therefore Z\left[\frac{1}{n!}\right] = e^{1/z}$. formula

8) $Z\left[\frac{1}{(n+1)!}\right] = Z\left[e^{1/z} - 1\right]$ $\quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

Here $x = \frac{1}{z}$.

9) $Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$

$= \sum_{n=0}^{\infty} \frac{a^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$

$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$

formula: $(1-x)^{-1} = 1 + x + x^2 + \cdots$ $x = \frac{a}{z}$.

$$Z[a^n] = \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \frac{1}{\left(1 - \frac{a}{z}\right)} = \frac{1}{\left(\frac{z-a}{z}\right)}$$

$$= \frac{z}{z-a}$$

$$Z[a^n] = \frac{z}{z-a}.$$