Reduce the quadratic form $2x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 = 0$ to canonical form by orthogonal reduction .Find rank, index, signature and nature

Step 1:

The matrix form is

$A = [2\ 2\ 0\ 2\ 2\ 0\ 0\ 0\ 1\ ]$

Step 2:

Characteristic equation ,Eigen values, Eigen vectors

C_1 =Sum of leading diagonal elements

=2+2+1 =5

C_2= Sum of minors of leading diagonal elements

=4

C_3=$|A|$

=$|2\ 2\ 0\ 2\ 2\ 0\ 0\ 0\ 1\ |$

= 0

The characteristic equation is

$\lambda^3 - 5\lambda^2 + 4\lambda = 0$

The eigen values are 0,1,4

The eigen vectors are $(A - \lambda I)X$=0

$$[(2\ 2\ 0\ 2\ 2\ 0\ 0\ 0\ 1\ ) - \lambda(1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ )]\left(x_1\ x_2\ x_3\right) = 0$$

$$(2 - \lambda\ 2\ 0\ 2\ 2 - \lambda\ 0\ 0\ 0\ 1 - \lambda)\left(x_1\ x_2\ x_3\right) = 0$$

$$(2\ 2\ 0\ 2\ 2\ 0\ 0\ 0\ 1)\left(x_1\ x_2\ x_3\right) = 0$$

CASE (i)

When $\lambda = 0$

$$(2\ 2\ 0\ 2\ 2\ 0\ 0\ 0\ 1)\left(x_1\ x_2\ x_3\right) = 0$$

The cofactor of first row elements are $(2\ \ -2\ 0)$ ie $(1\ \ -1\ 0)$

The Eigen vector when $\lambda = 0$ is $(1\ \ -1\ 0)$

CASE (ii)

When $\lambda = 1$

$$(1\ 2\ 0\ 2\ 1\ 0\ 0\ 0\ 0)\left(x_1\ x_2\ x_3\right) = 0$$

The cofactor of third row elements are $(0\ 0\ \ -3)$ ie $(0\ 0\ \ -1)$

The Eigen vector when $\lambda = 1$is $(0\ 0\ \ -1)$

CASE (iii)

When $\lambda = 4$

$$(1\ 2\ 0\ 2\ 1\ 0\ 0\ 0\ 0)\left(x_1\ x_2\ x_3\right) = 0$$

The cofactor of first row elements are $(6\ 6\ 0)$ ie $(1\ 1\ 0)$

The Eigen vector when $\lambda = 4$ is $(2\ \ -2\ 1)$

STEP 3:

To check pair wise orthogonality

$X_1^T X_2 = (2\ -2\ 0)(0\ 0\ -1) = 0$

$X_2^T X_3 = (0\ 0\ -1)(1\ 1\ 0) = 0$

$X_3^T X_1 = (1\ 1\ 0)(2\ -2\ 0) = 0$

STEP 4:

To find normalized vector

Eigen vector	l(x)=$\sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized vector = $\left(x_1/l(x_1)\ x_2/l(x_2)\ x_3/l(x_3)\right)$
$(1\ -1\ 0)$	$\sqrt{1+1+0}=\sqrt{2}$	$\left(\frac{1/}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}\ 0\right)$
$(0\ 0\ -1)$	$\sqrt{0+0+1}=\sqrt{1}$	$(0\ 0\ -1)$
$(1\ 1\ 0)$	$\sqrt{1+1+0}=\sqrt{2}$	$\left(\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 0\right)$

STEP 5:

Normalized modal matrix

N =$\left[\frac{1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\ 0\ -1\ 0\right]$

$N^T = \left[\frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}\ 0\ 0\ 0\ -1\ \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 1\right]$

STEP 6:

$NN^T = N^TN = I$

$N^TN = \left(\frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}\ 0\ 0\ 0\ -1\ \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 1\right)\left(\frac{1/}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\ 0\ -1\ 0\right)$

$= (1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1)$

$= I$

STEP 7:

To find diagonalyze matrix

$N^TAN = D$

$N^TA = \left(\frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}\ 0\ 0\ 0\ -1\ \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 1\right)(2\ 2\ 0\ 2\ 2\ 0\ 0\ 0\ 1)$

$= \left(0\ 0\ 0\ 0\ 0\ -1\ \frac{4}{\sqrt{2}}\ \frac{4}{\sqrt{2}}\ 0\right)$

$N^TAN = \left(0\ 0\ 0\ 0\ 0\ -1\ \frac{4}{\sqrt{2}}\ \frac{4}{\sqrt{2}}\ 0\right)\left(\frac{1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\ 0\ -1\ 0\right)$

$=(0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 4)$

$= D$

Step 8:

$Y^T DY = 0$

$0y_1^2 + y_2^2 + 4y_3^2 = 0$

The index p=2

Rank r=2

Signature s=2p-r =2

The nature is semi positive