PARTIAL DERIVATIVES:

Let $u = f(x,y)$ be a function of two independent variables. Differentiating 'u' w.r.to 'x' keeping 'y' as constant is known as partial derivative of u and w.r.to x and is denoted by $\frac{\partial u}{\partial x}$ (ie) u_x.

Similarly. $\frac{\partial u}{\partial y}$ (ie) u_y.

NOTE:

(i) If $u = v + w + \ldots$ where v, w are all functions of $x, y, z \ldots$ then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} + \cdots$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} + \cdots$$

ii) If u & v are functions of x, y, z etc then

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}.$$

$$\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}.$$

iii) If u & v are functions of x, y, z etc then.

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(iv) If u is a function of t where t is a function of the variables $x, y, z \ldots$ then.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} \quad \ldots$$

SUCCESSIVE PARTIAL DIFFERENTIATION:

Let $z = f(x,y)$ then $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ being the function of x & y can be further be differentiated partially w.r.to x & y.

$\therefore$ we have $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}$

Note: $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

(1) If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Soln: $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

Soln: $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

$$\frac{\partial u}{\partial x} = \frac{1}{y} - \frac{z}{x^2} \Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{y} - \frac{z}{x}.$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2} + \frac{1}{z} \Rightarrow y\frac{\partial u}{\partial y} = -\frac{x}{y} + \frac{y}{z}$$

$$\frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x} \Rightarrow z\frac{\partial u}{\partial z} = -\frac{y}{z} + \frac{z}{x}$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

(2) If $u = (x-y)^2 + (y-z)^2 + (z-x)^2$. p.T. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln: $u = (x-y)^2 + (y-z)^2 + (z-x)^2$.

$$\frac{\partial u}{\partial x} = 2(x-y)(1) + 2(z-x)\cdot(-1) = 2(x-y) - 2(z-x)$$

$$\frac{\partial u}{\partial y} = 2(y-z)(1) + 2(x-y)(-1) = 2(y-z) - 2(x-y)$$

$$\frac{\partial u}{\partial z} = 2(z-x)(1) + 2(y-z)\cdot(-1) = 2(z-x) - 2(y-z)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

5) If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ S.T. $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$.

Soln: $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$.

$$2r \cdot \frac{\partial r}{\partial x} = 2(x-a).$$

p.d.w.r.to x

$$\frac{\partial r}{\partial x} = \frac{(x-a)}{r}.$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - (x-a)\frac{\partial r}{\partial x}}{r^2} = \frac{r - (x-a)\cdot\frac{(x-a)}{r}}{r^2}$$

$$= \frac{r^2 - (x-a)^2}{r^3}$$

p.d.w.r.to y

$$\frac{\partial r}{\partial y} = \frac{y-b}{r}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - (y-b)^2}{r^3}$$

p.d.w.r.to z

$$\frac{\partial r}{\partial z} = \frac{z-c}{r}$$

$$\frac{\partial^2 r}{\partial z^2} = \frac{r^2 - (z-c)^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{r^2 - (x-a)^2 + r^2 - (y-b)^2 + r^2 - (z-c)^2}{r^3}$$

$$= \frac{3r^2 - [(x-a)^2 + (y-b)^2 + (z-c)^2]}{r^3} = \frac{3r^2 - r^2}{r^3}$$

$$= \frac{2}{r}$$