TOTAL DERIVATIVE

1) If $u = f(x,y)$ then total differential of u is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

2) If $u = f(x,y)$ where $x = g_1(t)$, $y = g_2(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

3) If $u = f(x,y)$ where y is a function of x then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \text{and}$$

$$\frac{du}{dy} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dy} + \frac{\partial u}{\partial y}$$

Differentiation of Implicit function :

If $f(x,y) = c$ where c may be zero or non-zero is an implicit function of x & y then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Type - I

1) If $u = x^2 + y^2$ then find total differential of u.

Soln: $u = x^2 + y^2$

$$\frac{\partial u}{\partial x} = 2x,$$

$$\frac{\partial u}{\partial y} = 2y.$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= 2x\, dx + 2y\, dy.$$

Type - II

(3) Find $\frac{du}{dt}$ if $u = x^2 + y^2 + z^2$ where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$.

Soln: $u = x^2 + y^2 + z^2$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z.$$

Gn: $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

$$\frac{dx}{dt} = e^t \quad \frac{dy}{dt} = e^t \cos t + e^t \sin t \quad \frac{dz}{dt} = -e^t \sin t + e^t \cos t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

$$= 2x\cdot e^t + 2y\cdot(e^t\cos t + e^t\sin t) + 2z\cdot(e^t\cos t - e^t\sin t)$$

$$= 2e^t e^t + 2e^t\sin t\,(e^t\cos t + e^t\sin t) + 2e^t\cos t\,(e^t\cos t - e^t\sin t)$$

$$= 2e^{2t} + \cancel{2e^{2t}\sin t\cos t} + 2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t - \cancel{2e^{2t}\cos t\sin t}$$

$$= 2e^{2t} + 2e^{2t} = 4e^{2t}.$$

Type -

(iv) If $x^3 + y^3 = 3axy$ find $\frac{dy}{dx}$

Soln: $x^3 + y^3 - 3axy = 0$

Let $f(x,y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{3(x^2 - ay)}{3(y^2 - ax)} = \frac{ay - x^2}{y^2 - ax}.$$