Curvilinear Motion

When the path described by a moving particle is a curved line, then the motion of the particle is known as curvilinear motion of translation. If the continuous path described by the moving particle remains confined to a plane, then it is known as 'plane motion' ie, in plane motion, the motion of a particle is in two dimensions. If the path described by the particle does not lie in one plane, then it is known as 'spatial motion'. In this chapter, we will see only the plane motion.

When the motion of a particle in a plane is considered, single variable is not sufficient but two variables (x and y) ie, two co- ordinates are to be specified to locate the particle at any instant. For the location of a particle along a curved path, following two systems are followed.

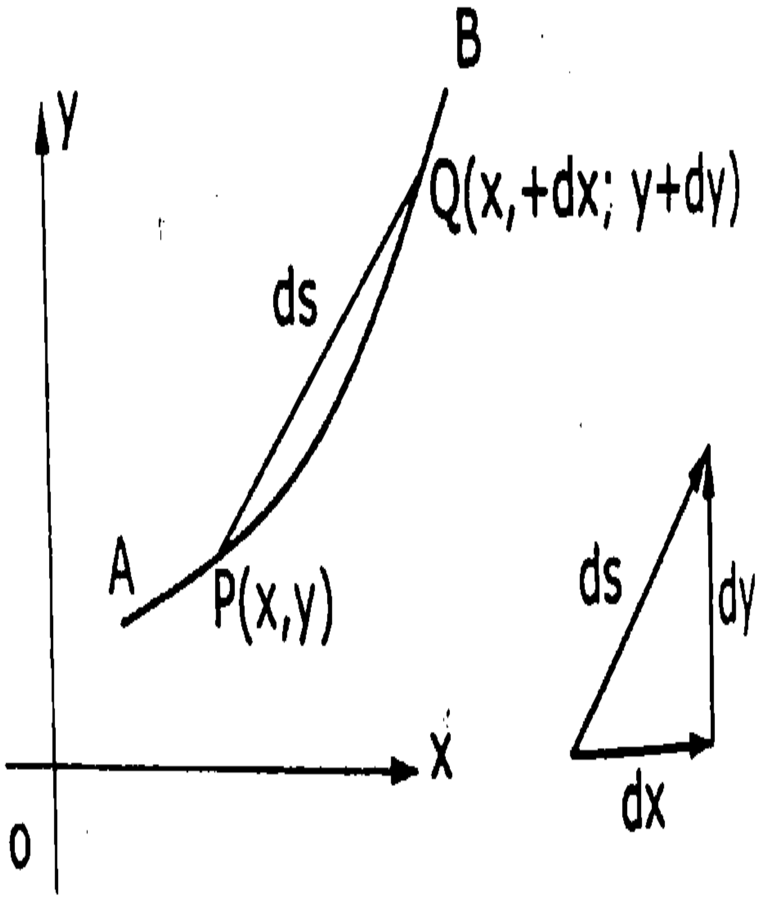
(i) Cartesian system (or rectangular co-ordinates)

(ii) Polar system (or radial co-ordinates)

In this chapter, cartesian system is followed to locate a particle on the curved path.

Horizontal component of velocity (parallel to OX), $V_x = \underset{\partial t \to 0}{Lt} \frac{\partial x}{\partial t}$ or $\boxed{V_x = \frac{dx}{dt}}$

|||ly Vertical component of velocity (parallel to OY), $V_y = \underset{\partial t \to 0}{Lt} \frac{\partial y}{\partial t}$ or $\boxed{V_y = \frac{dy}{dt}}$



∴ The resultant velocity of the particle at any point, $V = \sqrt{(V_x)^2 + (V_y)^2}$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

and the angle of inclination of resultant velocity with x axis, $\boxed{\alpha = \tan^{-1}\left(\frac{V_y}{V_x}\right)}$

Acceleration of the particle

Let the velocity of the particle changes from P (x,y) to Q(x+dx; y + dy) in a small interval ∂t.

$\therefore$ Acceleration of the particle along OX, $a_x = \underset{\partial t \to 0}{Lt} \frac{\partial V_x}{\partial t} = \frac{dV_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

$|||^{ly}$ Acceleration of the particle along OY, $a_y = \underset{\partial t \to 0}{Lt} \frac{\partial V_y}{\partial t} = \frac{dV_y}{dt}$

$$= \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$\therefore$ Resultant acceleration, $a = \sqrt{(a_x)^2 + (a_y)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$

Let the angle of inclination of acceleration with x axis is ϕ, then, $\tan\phi = \frac{a_y}{a_x}$

$$\therefore \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right)$$

Example 1 :

The motion of a particle along a curved path is given by the equations.

$$x = t^2 + 8t + 4, \quad \text{and} \quad y = t^3 + 3t^2 + 8t + 4$$

determine

(i) Initial velocity of the particle *(ii) velocity of the particle at t = 2 sec*

(iii) acceleration of the particle at t = 0 and

(iv) acceleration of the particle at t = 2 sec

Solution

Given, $x = t^2 + 8t + 4$ and $y = t^3 + 3t^2 + 8t + 4$

Velocity components of the particle.

Horizontal component of velocity, $V_x = \frac{dx}{dt} = \frac{d}{dt}(t^2 + 8t + 4)$

$$= 2t + 8 \quad \ldots \text{(i)}$$

Vertical component of velocity, $V_y = \frac{dy}{dt} = \frac{d}{dt}(t^3 + 3t^2 + 8t + 4)$

$$= 3t^2 + 6t + 8 \quad \ldots \text{(ii)}$$

Acceleration components of the particle

Horizontal component of acceleration, $a_x = \frac{d}{dt}(V_x) = \frac{d}{dt}(2t + 8) = 2$ ... (iii)

Vertical component of acceleration, $a_y = \frac{d}{dt}(V_y) = \frac{d}{dt}(3t^2 + 6t + 8) = 6t + 6$ ... (iv)

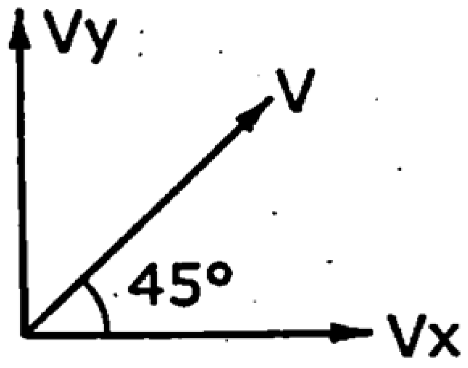
(i) Initial velocity of the particle

Put t = 0 in equations (i) and (ii).

$$V_x = 2t + 8 = 2(0) + 8 = 8 \text{m/s}$$

and $V_y = 3t^2 + 6t + 8$

$$= 3(0) + 6(0) + 8 = 8 \text{m/s}$$



$$\therefore \text{Velocity, } V = \sqrt{V_x^2 + V_y^2} = \sqrt{8^2 + 8^2} = 11.31 \text{ m/s}$$

$$\text{and } \theta = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \tan^{-1}\left(\frac{8}{8}\right) = 1 = 45°$$

(ii) Velocity at t = 2sec

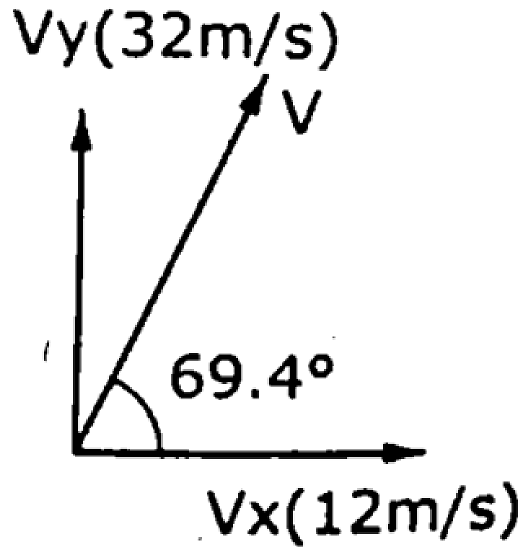
Put t = 2 in equations (i) and (ii)

$$V_x = 2t + 8 = 2(2) + 8 = 12 \text{ m/s}$$

and $$V_y = 3t^2 + 6t + 8 = 3(2)^2 + 6(2) + 8 = 32 \text{ m/s.}$$

$\therefore$ Velocity, 'V' at 2 sec $= \sqrt{V_x^2 + V_y^2}$

$$= \sqrt{12^2 + 32^2} = 34.17 \text{ m/s.}$$



$$\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$

$$= \tan^{-1}\left(\frac{32}{12}\right) = 69.4°$$

substitute $t = 0$ in equations (iii) and (iv).

$$a_x = 2 \text{ m/s}^2$$

$$a_y = 6t + 6 = 6(0) + 6 = 6 \text{ m/s}^2$$

$\therefore$ acceleration at $t = 0$, $a = \sqrt{(a_x)^2 + (a_y)^2} = \sqrt{2^2 + 6^2} = 6.324 \text{ m/s}^2$

Let ϕ be the angle of inclination of 'a' with horizontal. then, $\tan\phi = \dfrac{a_y}{a_x}$

$$\therefore \quad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{6}{2}\right) = 71.56^\circ$$

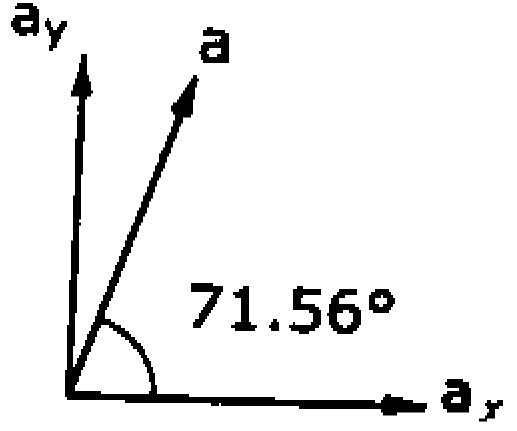
(iv) acceleration at t = 2 sec

substitute t = 2 in equations (iii) and (iv)

$$a_x = 2 \text{ m/s}^2$$

$$a_y = 6t + 6$$

$$= 6(2) + 6 = 18 \text{ m/s}^2$$

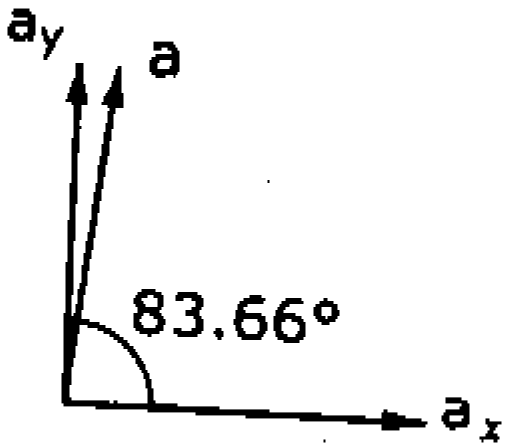


$\therefore$ acceleration at t= 2 sec, $a = \sqrt{(a_x)^2 + (a_y)^2}$

$$= \sqrt{2^2 + 18^2} = 18.11 \text{ m/s}^2$$

angle of inclination of acceleration with horizontal,

$$\phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{18}{2}\right) = 83.66°$$



Example 2:

> *The motion of a particle is specified by the equations*
>
> $$x = 5t + 0.75\,t^2 \quad \text{and} \quad y = 4t + 0.6\,t^2$$
>
> *where x and y are in metres and t is in seconds. Determine*
>
> *(i) the path of the particle* *(ii) velocity of the particle after 4 sec*
>
> *(iii) acceleration of the particle after 2 sec*

Solution

Given, $x = 5t + 0.75t^2$ ..... (i)

$y = 4t + 0.6t^2$ ..... (ii)

(i) Path of the particle

To find the path of the particle, eliminate 't' from both the equations.

equation (i) × 4 ⇒ $4x = 20t + 3t^2$ ..... (iii)

equation (ii) × 5 ⇒ $5y = 20t + 3t^2$ ..... (iv)

Equation (iii) – Equation (iv) $\Rightarrow (4x - 5y) = 0$

or $\quad 4x = 5y \quad$ or $\quad y = \frac{4x}{5} \quad$ or $\quad y = 0.8x$

$y = 0.8x$ is in the form of $y = mx + c$, which is the equation of straight line.

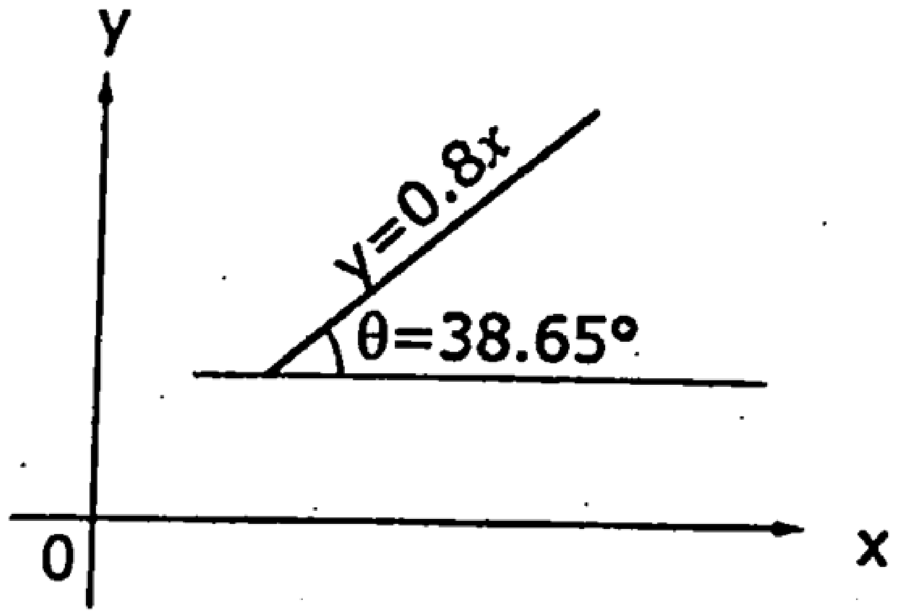
in which, $\quad c = 0$;

$\quad m =$ slope of straight line $= 0.8$

or $\quad \tan\theta = 0.8$

$\therefore \quad \theta = \tan^{-1}(0.8) = 38.65°$

where, θ is the angle of inclination of the straight line with horizontal. The path is shown in fig. 17.2



(ii) Velocity after 4 sec

Horizontal component of velocity,

$$V_x = \frac{dx}{dt} = \frac{d}{dt}(5t + 0.75t^2) = 5 + 1.5t \quad ..... \quad (i)$$

Vertical component of velocity,

$$V_y = \frac{dy}{dt} = \frac{d}{dt}(4t + 0.6t^2) = 4 + 1.2t \quad \text{..... (ii)}$$

To find the velocity after 4 sec, substitute t = 4 in equations (i) and (ii).

$$V_x = 5 + 1.5t = 5 + (1.5 \times 4) = 11 \text{ m/s}$$

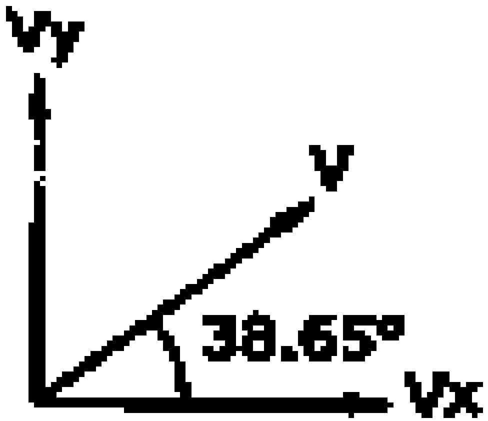
$$V_y = 4 + 1.2t = 4 + (1.2 \times 4) = 8.8 \text{ m/s}$$

$$\therefore \text{ Velocity at 4 sec, } V_4 = \sqrt{V_x^2 + V_y^2}$$

$$= \sqrt{11^2 + 8.8^2} = 14.08 \text{ m/s}$$

$$\text{angle of velocity with horizontal, } \theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$

$$= \tan^{-1}\left(\frac{8.8}{11}\right) = 38.65°$$



(iii) Acceleration after 2 sec

Horizontal component of acceleration, $a_x = \frac{d}{dt}(V_x) = \frac{d}{dt}(5 + 1.5t) = 1.5 \text{ m/s}^2$

Vertical component of acceleration, $a_y = \frac{d}{dt}(V_y) = \frac{d}{dt}(4 + 1.2t) = 1.2 \text{ m/s}^2$

$\therefore$ acceleration, $a = \sqrt{(a_x)^2 + (a_y)^2} = \sqrt{1.5^2 + 1.2^2} = 1.92 \text{ m/s}^2$

Example 3:

> *The motion of a body moving on a curved path is given by the equations, $x = 4\sin 3t$ and $y = 4\cos 3t$. Find the velocity and acceleration after 2 seconds.*

Solution

Given, $x = 4\sin 3t$ and

$y = 4\cos 3t$

Velocity of the particle

$$V_x = \frac{dx}{dt} = \frac{d}{dt}(4\sin 3t) = 12\cos 3t$$

$$V_y = \frac{dy}{dt} = \frac{d}{dt}(4\cos 3t) = -12\sin 3t$$

$$\therefore \text{Resultant velocity, } V = \sqrt{V_x^2 + V_y^2}$$

$$= \sqrt{(12\cos 3t)^2 + (-12\sin 3t)^2} \qquad (\because \sin^2 3t + \cos^2 3t = 1)$$

$$= \sqrt{12^2(\cos^2 3t + \sin^2 3t)}$$

$$= \sqrt{12^2} = 12 \text{ m/s}$$

Velocity of the particle after 2 sec

From the above result, the velocity of the particle at any time interval is constant. ie, 12 m/s.

Acceleration of the particle

$$a_x = \frac{d}{dt}(V_x) = \frac{d}{dt}(12\cos 3t) = -36\sin 3t$$

$$a_y = \frac{d}{dt}(V_y) = \frac{d}{dt}(-12\sin 3t) = -36\cos 3t$$

$$\therefore \text{acceleration, } a = \sqrt{a_x^2 + a_y^2}$$

$$= \sqrt{(-36\sin 3t)^2 + (-36\cos 3t)^2}$$

$$= \sqrt{36^2(\sin^2 3t + \cos^2 3t)} \qquad (\because \sin^2 3t + \cos^2 3t = 1)$$

$$= \sqrt{36^2} = 36 \text{ m/s}^2$$

Acceleration of the particle after 2sec

From the above result, the acceleration of the particle at any time is constant of 36 m/s^2 ('a' is not in 't' term, this shows, it is a constant measure at any time).

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<th>Cartesian Co-ordinate System</th>
<th>Polar System</th>
<th>Normal and Tangential System</th>
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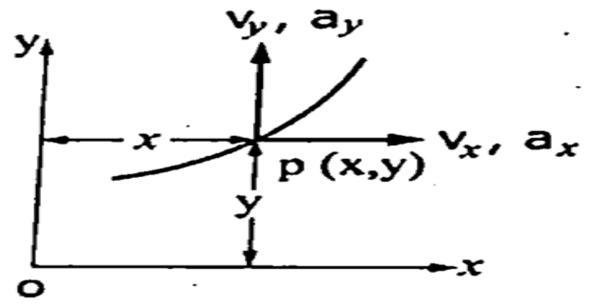

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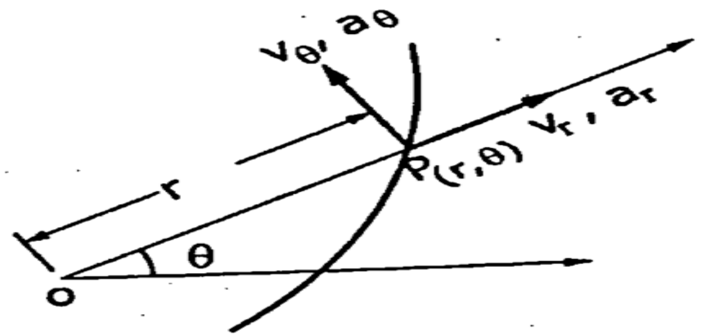

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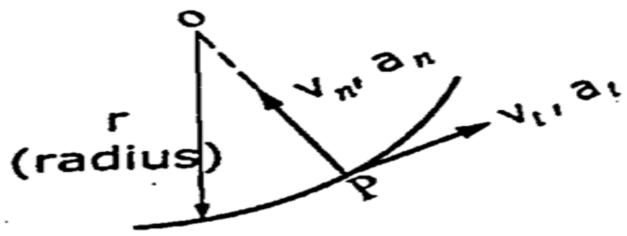

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Velocity components

Velocity along X direction,

$$V_x = \frac{dx}{dt}$$

|||ly Velocity along y direction, $V_y = \frac{dy}{dt}$

Resultant velocity,

$$V = \sqrt{(V_x)^2 + (V_y)^2}$$

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Velocity Components

Radial velocity, $V_r = \frac{dr}{dt}$

Transverse velocity,

$$V_\theta = r\frac{d\theta}{dt}$$

Resultant velocity,

$$a = \sqrt{(V_r)^2 + (V_\theta)^2}$$

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Velocity components

V = Velocity along the curve

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Acceleration Components

$$a_x = \frac{d\,V_x}{dt} = \frac{d^2x}{dt^2}$$

$$a_y = \frac{d\,V_y}{dt} = \frac{d^2y}{dt^2}$$

Resultant acceleration,

$$a = \sqrt{(a_x)^2 + (a_y)^2}$$

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Acceleration Components

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$a_\theta = \frac{1}{r}\frac{d}{dt}\left(r\frac{d^2\theta}{dt^2}\right)$$

Resultant acceleration,

$$a = \sqrt{(a_r)^2 + (a_\theta)^2}$$

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Acceleration Components

Normal acceleration, $a_n = \frac{V^2}{r}$

Tangential acceleration,

$$a_t = \frac{dV}{dt}$$

Resultant acceleration,

$$a = \sqrt{(a_n)^2 + (a_t)^2}$$

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A particle moves along a curve and its position is defined by, $r = 2\theta$ and $\theta = \frac{t^2}{2}$, where t is in sec, and r in metres. Determine the velocity and acceleration of the particle at $\theta = \frac{\pi}{4}$

Solution :

Given, $r = 2\theta$ and $\theta = \frac{t^2}{2}$

Now, $\frac{d\theta}{dt} = t$ and $\frac{d^2\theta}{dt^2} = 1$

$$r = 2\theta = 2\left(\frac{t^2}{2}\right) = t^2$$

$$\therefore \frac{dr}{dt} = 2t \quad \text{and} \quad \frac{d^2r}{dt^2} = 2$$

Velocity and acceleration components are required when $\theta = \frac{\pi}{4}$

It is given, $\theta = \frac{t^2}{2}$ or $t = \sqrt{2\theta}$

when $\theta = \frac{\pi}{4}$; $t = \sqrt{2 \times \pi/4} = 1.253$ sec.

Velocity components at t = 1.253 sec.

Radial velocity, $V_r = \frac{dr}{dt} = 2(1.253) = 2.506 \text{ m/s}$

Transverse velocity, $V_\theta = r\frac{d\theta}{dt} = t^2(t)$

$= t^3 = (1.253)^3 = 1.967 \text{ m/s}$

∴ Resultant Velocity, $V = \sqrt{(V_r)^2 + (V_\theta)^2}$

$= \sqrt{(2.506)^2 + (1.967)^2}$

$=$ **3.186 m/s (Ans)**

Acceleration components when t = 1.253 sec

Radial acceleration, $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$= 2 - \{ t^2 \times (t)^2 \} = 2 - (t^4)$$

$$= 2 - (1.253)^4 = 0.464 \text{ m/s}^2$$

Transverseacceleration, $a_\theta = \frac{2\,dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

$$= \{ 2\,(2t)\,(t) \} + \{ t^2\,(1) \}$$

$$= 4t^2 + t^2$$

$$= 5t^2 = 5(1.253)^2 = 7.85 \text{ m/s}^2$$

$\therefore$ Resultant acceleration, $a = \sqrt{(a_r)^2 + (a_t)^2} = \sqrt{(-0.464)^2 + (7.85)^2}$

$$= \mathbf{7.863\ m/s^2} \qquad \textbf{(Ans)}$$