13.1 Introduction:

In article 5.2.1. we have measured the moment of a force about a point as the product of its magnitude and the perpendicular distance between the line of action of the force and the point about which the force causing rotation.

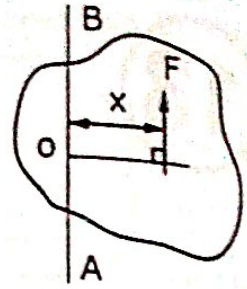


Fig. 13.1

Refering the fig 13.1, the moment of the force F about O, on AB axis, $M_o = F \times x$

This moment is also called as the first moment of the force about O, Let it be MO_1.

If this moment is again multiplied by x, then we get the moment of moment of the force, or second moment of the force. Let it be MO_2. **The second moment of the force, MO_2 is also called as the moment of inertia.**

Moment of Inertia MO_2 = First moment × distance

$$= MO_1 \times x$$

$$= (F \times x) \times x$$

$$= Fx^2$$

The moment of inertia is briefly written as M.I. and denoted by the symbol I.

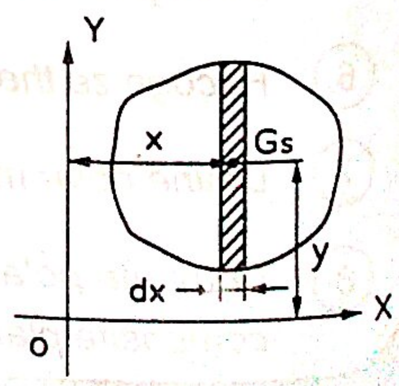
But, sometimes plane and solid figures are taken into consideration, instead of the force, and the second moment (or moment of inertia) are determined. In this chapter we will see the moment of inertia of plane figures.

13.2 Moment of Inertia of plane figures

Moment of inertia of a plane figure is generally called as 'area moment of inertia'. In SI system of units, units of area moment of inertia are mm^4, cm^4, m^4. **The moment of inertia is denoted by I and carries with it the symbol of the axes about which it is calculated. Thus the moment of inertia about an axis AB is denoted by I_{AB}. The moment of inertia about centroidal axes are denoted by I_{xx} and I_{yy}.** (Sometimes, the moment of inertia of a body about an axis passing through the centre of gravity of the body is denoted by I_{CG} or simply I_G).

Again, the moment of inertia of simple and composite plane figure are determined separately as in the case of centre of gravity. **The M.I. of simple plane figures are determined by the method of integration and the M.I of composite plane figures are determined by applying the theorems of moment of inertia.**

13.2.1 Moment of Inertia of simple plane figure by Integration



Consider a plane figure, whose moment of inertia is required about OX and OY axes, as shown in fig 13.2.

The plane figure may be split up into an infinite number of elements. Consider a strip of thickness 'dx' as shown in fig, its centroid being G_s.

13.2.3. Parallel axis theorem

It states that "the moment of inertia of a lamina about any axis in the plane of lamina is equal to the sum of the moment of inertia about a parallel centroidal axis in the plane of lamina and the product of the area of the lamina and square of the distance between the two axes".

In previous article, we arrived the M.I of a rectangle about its horizontal centroidal axis,

$$I_{XX} = \frac{bh^3}{12}$$

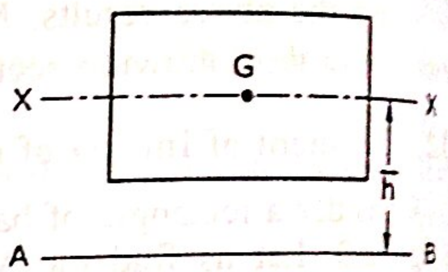


Fig. 13.4

The moment of inertia of the same rectangle, about any axis, but parallel to XX axis can be determined by parallel axis theorem. Let AB is a parallel axis, parallel to XX, at a distance of $\bar{h}$.

From parallel axis theorem,

$$I_{AB} = I_{XX} + A\bar{h}^2$$

using the above result, the M.I. of rectangle about its bottom edge can be determined as below

$$I_{AB} = I_{XX} + A(\bar{h})^2 \qquad \text{(Area, A = bh)}$$

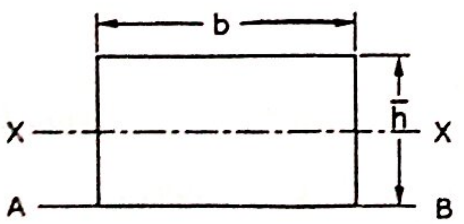


Fig. 13.5.

$$= \frac{bh^3}{12} + \left[(bh) \times \left(\frac{h}{2}\right)^2\right]$$

$$= \frac{bh^3}{12} + \left[\frac{bh^3}{4}\right] = \frac{4bh^3}{12}$$

$$= \frac{bh^3}{3}$$

13.2.6 Perpendicular Axis theorem.

It states that "If I_{OX} and I_{OY} be the moment of inertia of a lamina about two mutually perpendicular axes OX and OY in the plane of the lamina and I_{OZ} be the moment of inertia of the lamina about an axis normal to the lamina and passing through the point of intersection of the axes OX and OY, then

$$I_{OZ} = I_{OX} + I_{OY}$$

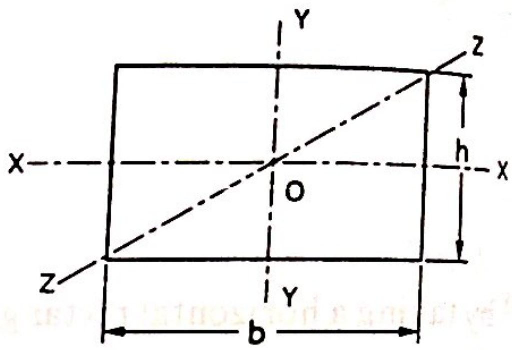


Fig. 13.11

For example, consider a rectangle of base 'b' and height 'h' as shown in fig 13.11.

We know, $I_{xx} = \frac{bh^3}{12}$ and $I_{yy} = \frac{hb^3}{12}$

From perpendicular axis theorem, M.I of the rectangle about the axis zz, passing through the point of intersection of xx and yy axes and normal to the plane of rectangle,

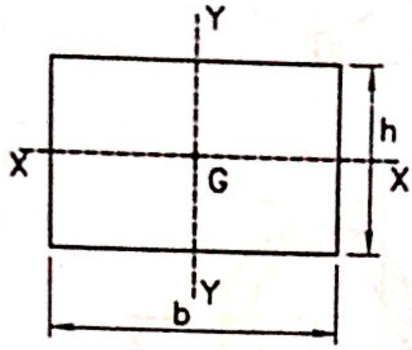
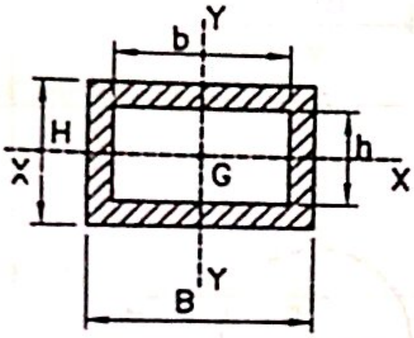
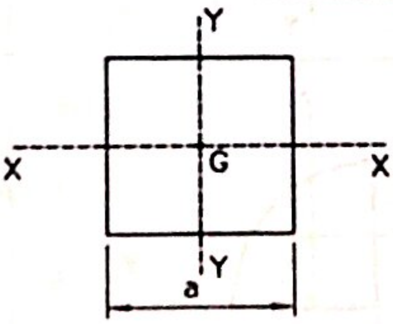
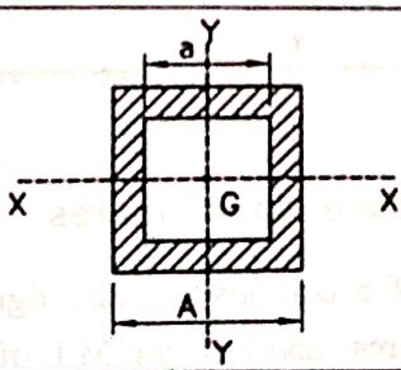
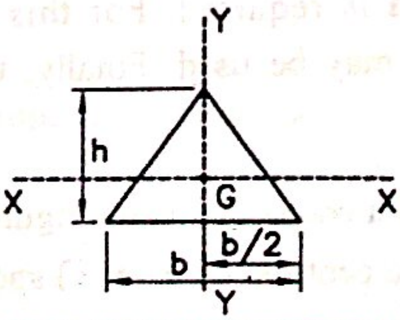
$$I_{zz} = I_{xx} + I_{yy}$$

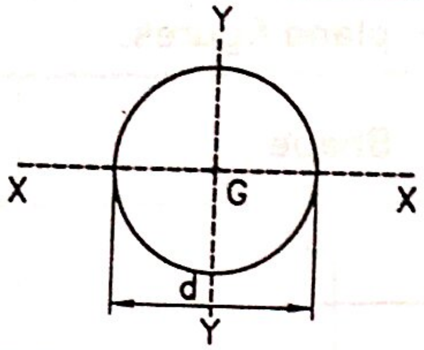
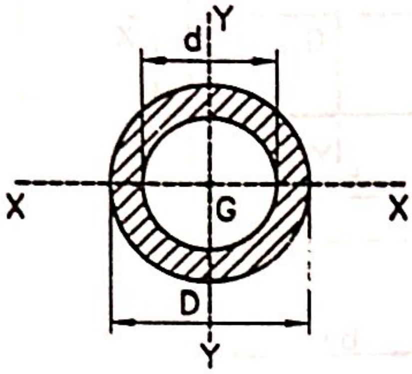
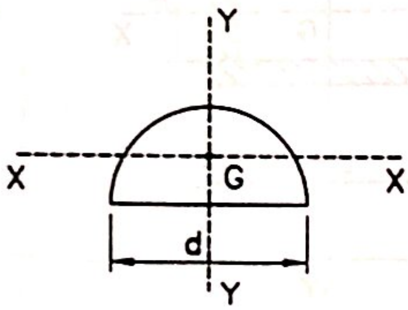
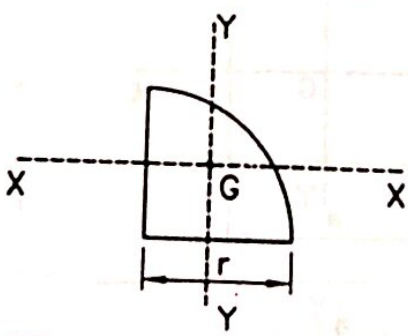
$$= \frac{bh^3}{12} + \frac{hb^3}{12}$$

$$\therefore\ I_{zz} = \frac{1}{12}[bh^3 + hb^3]$$

with the help of the perpendicular axis theorem, the M.I. of a circle is arrived as below.

Table 13.1 M.I. of common simple plane figures.

Sl. No	Name	Shape	I_{xx}	I_{yy}
1	Rectangle		$\frac{bh^3}{12}$	$\frac{hb^3}{12}$
2	Hollow rectangle		$\frac{1}{12}(BH^3 - bh^3)$	$\frac{1}{12}(HB^3 - hb^3)$
3	Square		$\frac{a^4}{12}$	$\frac{a^4}{12}$
4	Hollow Square		$\frac{1}{12}(A^4 - a^4)$	$\frac{1}{12}(A^4 - a^4)$
5	Triangle		$\frac{bh^3}{36}$	$\frac{hb^3}{48}$

6	Circle		$\frac{\pi d^4}{64}$	$\frac{\pi d^4}{64}$
7	Hollow Circle		$\frac{\pi}{64}(D^4 - d^4)$	$\frac{\pi}{64}(D^4 - d^4)$
8	Semi-circle		$0.0068d^4$	$\frac{\pi d^4}{128}$
9	Quadrant		$0.055r^4$	$0.055r^4$

Find the moment of inertia of a T section of flange 100 mm × 30 mm and web 20 mm × 80 mm about its centroidal axes.

Solution:

The T-section is drawn as shown in fig. 13.15.

The moment of inertia about the centroidal axes are required. Hence, we have to draw the centroidal axes XX and YY first, passing through the centroid of the section. The T-section is symmetrical about y axis, hence the yy axis can be drawn (the symmetrical axis) directly.

But to draw the xx axis, we need $\bar{y}$. hence $\bar{y}$ is to be determined first.

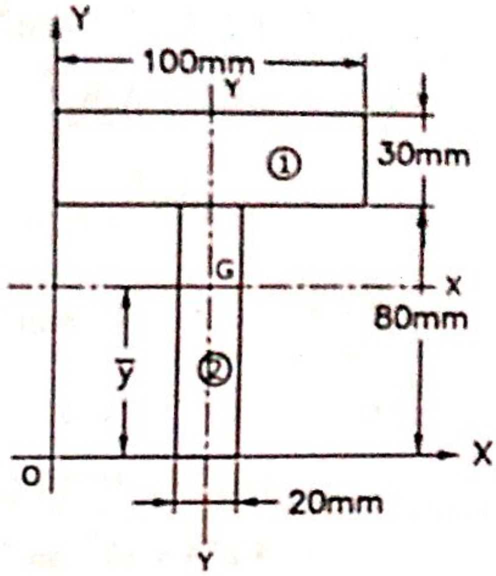


Fig. 13.15

To find $\bar{y}$

The T-section is divided into two components.

Portion (1) : (Flange 100 mm × 30 mm)

area, $a_1 = 100 \times 30 = 3000 \text{ mm}^2$; $\quad y_1 = 80 + \frac{30}{2} = 95 \text{ mm}$

Portion (2) : (Web 20 mm × 80 mm)

area, $a_2 = 20 \times 80 = 1600 \text{mm}^2$; $\quad y_2 = \frac{80}{2} = 40 \text{ mm}$

using the relation, $\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(3000 \times 95) + (1600 \times 40)}{3000 + 1600} = 75.87 \text{ mm}$

Hence, the horizontal centroidal axis xx, is drawn at a height of $\bar{y}$ (75.87mm) from the bottom of the web (ie., the horizontal reference axis ox).

Moment of Inertia about xx axis.

M.I. of T-section about xx axis is equal to the sum of the M.I. of the components (1) and (2) about the same axis xx.

ie, $I_{xx} = (I_{xx})_1 + (I_{xx})_2$

now to find the M.I. of component (1) about xx axis, apply parallel axis theorem.

$$(I_{xx})_1 = (I_G)_1 + A_1 \bar{h}_1^2$$

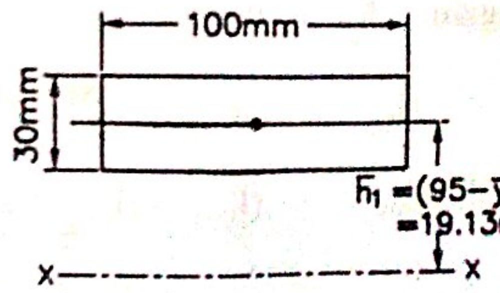


Fig. 13.16

where $(I_G)_1$ is I_{xx} of component (1)

$$\therefore \quad (I_{xx})_1 = \left(\frac{100 \times 30^3}{12}\right) + [(100 \times 30) \times 19.13^2)]$$

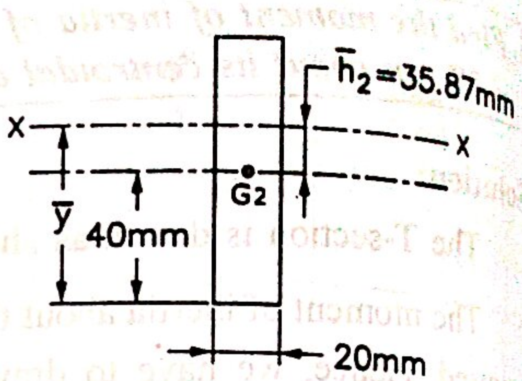


Fig. 13.17

$$= 1.323 \times 10^6 \text{ mm}^4$$

$|||^{ly}$ $\quad (I_{xx})_2 = (IG)_2 + A_2\bar{h}_2^{\;2}$

$$= \frac{20 \times 80^3}{12} + [(20 \times 80) \times (35.87)^2]$$

$$= 2.911 \times 10^6 \text{ mm}^4$$

$$\therefore \quad I_{xx} = (I_{xx})_1 + (I_{xx})_2$$

$$= (1.323 \times 10^6) + (2.911 \times 10^6)$$

$$= 4.234 \times 10^6 \text{ mm}^4$$

Moment of Inertia about yy axis

$$I_{yy} = (I_{yy})_1 + (I_{yy})_2$$

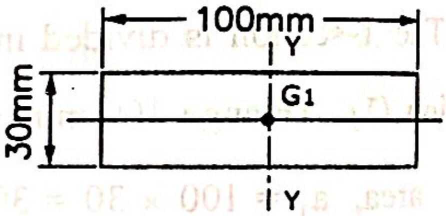


Fig. 13.18

$$(I_{yy})_1 = (I_G)_1 + A_1\bar{h}_1^{\;2}$$

here $(I_G)_1 = (I_{yy})$ of component (1)

$\bar{h}_1 = 0$ as yy axis of the composite section, lies on the yy of the component (1)

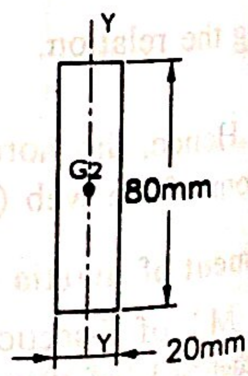


Fig. 13.19

$$\therefore (I_{yy})_1 = (IG)_1$$

$$= \frac{30 \times 100^3}{12} = 2.5 \times 10^6 \text{ mm}^4$$

$|||^{ly}$ $(I_{yy})_2 = (I_G)_2 + A_2\bar{h}_2^2$

again $\bar{h}_2 = 0$

$$\therefore \quad (I_{yy})_2 = (IG)_2 = \frac{80 \times 20^3}{12} = 5.33 \times 10^4 \text{ mm}^4$$

$$\therefore (I_{yy}) = (I_{yy})_1 + (I_{yy})_2 = (2.5 \times 10^6) + (5.33 \times 10^4) = 2.553 \times 10^6 \text{ mm}^4$$

$$\therefore \mathbf{I_{xx} = 4.233 \times 10^6 \text{ mm}^4}$$
$$\mathbf{I_{yy} = 2.553 \times 10^6 \text{ mm}^4}$$

Find the moment of inertia of an un symmetrical I section shown in fig 13.20 about its centroidal axes.

Solution

Due to symmetry, $\bar{x} = \frac{100}{2} = 50$ mm

To find $\bar{y}$

Divide the I section into three portions.

Portion 1 (Top flange , 60 mm × 20 mm)

area, $a_1 = 60 \times 20 = 1200 mm^2$

$y_1 = 20 + 60 + \frac{20}{2} = 90$ mm

Portion (2) : (Web , 20 mm × 60 mm)

area, $a_2 = 20 \times 60 = 1200 mm^2$

$y_2 = 20 + \frac{60}{2} = 50$ mm

Portion (3) (Bottom Flange ,
100 mm × 20 mm)

area, $a_3 = 100 \times 20 = 2000 mm^2$

$y_3 = \frac{20}{2} = 10$ mm

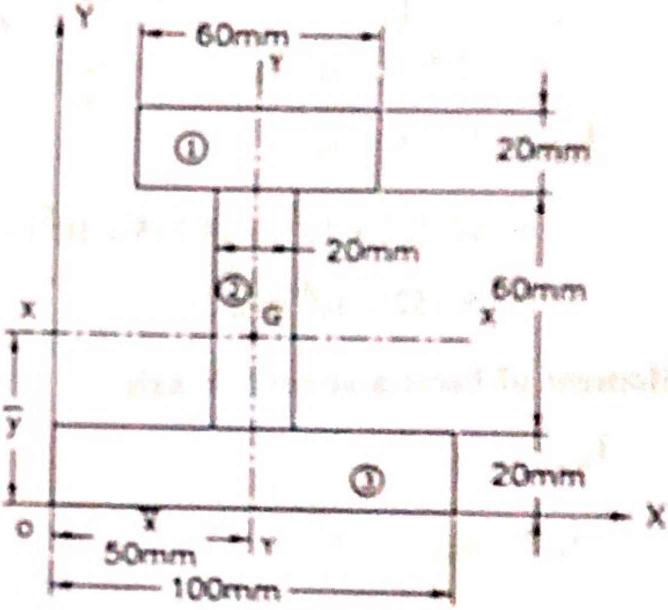


Fig 13.20

$$\therefore \bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3} = \frac{(1200 \times 90) + (1200 \times 50) + (2000 \times 10)}{1200 + 1200 + 2000}$$

$= 42.72$ mm

∴ Horizontal centroidal axis XX is at a height of 42.72 mm from the horizontal reference axis OX.

Moment of Inertia about XX axis

$$I_{XX} = I_1 + I_2 + I_3$$

where I_1, I_2 and I_3 are the moment of inertia of components 1, 2 and 3 respectively about XX axis of I section.

$$I_1 = I_{G_1} + A_1\bar{h}_1^2$$

$$= \frac{60 \times 20^3}{12} + [(60 \times 20) \times 47.28^2]$$

$$= 2.722 \times 10^6 \text{ mm}^4$$

$$\bar{h}_1 = \bar{y} \sim y_1 = 42.72 \sim 90 = 47.28 \text{ mm}$$

$$I_2 = I_{G_2} + A_2\bar{h}_2^2$$

$$= \frac{20 \times 60^3}{12} + [(20 \times 60) \times 7.28^2]$$

$$= 4.235 \times 10^5 \text{ mm}^4$$

$$\bar{h}_2 = \bar{y} \sim y_2 = 42.72 \sim 50 = 7.28 \text{ mm}$$

$$I_3 = I_{G_3} + A_3\bar{h}_3^2$$

$$= \frac{100 \times 20^3}{12} + [(100 \times 20) \times 32.72^2]$$

$$= 2.207 \times 10^6 \text{ mm}^4$$

$$\bar{h}_3 = \bar{y} \sim y_3 = 42.72 \sim 10 = 32.72 \text{ mm}$$

$$\therefore I_{XX} = I_1 + I_2 + I_3$$

$$= (2.722 \times 10^6) + (4.235 \times 10^5) + (2.207 \times 10^6)$$

$$= 5.352 \times 10^6 \text{ mm}^4$$

Moment of Inertia about YY axis

$$I_{YY} = I_1 + I_2 + I_3$$

$$I_1 = I_{G1} + A_1\bar{h}_1^2$$

here, $I_{G1} = (I_{yy})$ self of (1)

$$= \frac{20 \times 60^3}{12} + [(20 \times 60) \times 0]$$

$$h_1 = \bar{x} \sim x_1 = 0 \quad (\because \bar{x} = x_1)$$

$$= 3.6 \times 10^5 \text{ mm}^4$$

$$I_2 = I_{G2} + A_2\bar{h}_2^2$$

$$\bar{h}_2 = \bar{x} \sim x_2 = 0 \quad (\because \bar{x} = x_2)$$

$$= \frac{60 \times 20^3}{12} + [(60 \times 20) \times 0]$$

$$= \frac{60 \times 20^3}{12} = 4 \times 10^4 \text{ mm}^4$$

$$I_3 = I_{G3} + A_3 h_3^2 \qquad h_3 = \bar{x} - x_3$$

$$= 0$$

$$(\because \bar{x} = x_3)$$

$$= \frac{20 \times 100^3}{12} + [(20 \times 100) \times 0]$$

$$= \frac{20 \times 100^3}{12} = 1.667 \times 10^6 \text{ mm}^4$$

$$\therefore I_{yy} = I_1 + I_2 + I_3$$

$$= (3.6 \times 10^5) + (4 \times 10^4) + (1.667 \times 10^6)$$

$$= 2.067 \times 10^6 \text{mm}^4$$

$$\therefore \mathbf{I_{xx} = 5.352 \times 10^6 \text{ mm}^4}$$

$$\mathbf{I_{yy} = 2.067 \times 10^6 \text{ mm}^4}$$