# MAXIMA & MINIMA OF FUNCTIONS OF TWO VARIABLES

Conditions for $f(x,y)$ to be maximum (or) minimum

(i) Necessary condition:

The necessary condition for $f(x,y)$ to have a maximum or minimum at the point $(a,b)$ are $\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = 0$ & $\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = 0$

(ii) Sufficient condition:

If $f_x(a,b)=0$, $f_y(a,b)=0$, $f_{xx}(a,b)=A$, $f_{xy}(a,b)=B$, $f_{yy}(a,b)=C$ then.

(1) $f(a,b)$ is maximum value if $AC-B^2>0$ & $A<0$ and the point $(a,b)$ is called the maximum point.

(2) $f(a,b)$ is minimum value if $AC-B^2>0$ & $A>0$ and the point $(a,b)$ is called the minimum point.

(3) $f(a,b)$ is neither maximum nor minimum

(a) not an extremum if $AC-B^2<0$ and the point $(a,b)$ is called saddle point.

(4) If $AC-B^2=0$ then the test is inconclusive.

## Stationary points:-

A function $f(x,y)$ is said to be stationary at the point $(a,b)$ if $f_x = 0$, $f_y = 0$

**Notations:**

$$\frac{\partial f}{\partial x} = f_x;\ \frac{\partial f}{\partial y} = f_y;\ \frac{\partial^2 f}{\partial x^2} = f_{xx};\ \frac{\partial^2 f}{\partial y^2} = f_{yy};\ \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

1) Find the maximum and minimum value of $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Soln:

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$$f_x = 3x^2 + 3y^2 - 6x$$

$$f_y = 6xy - 6y$$

$$A:\ f_{xx} = 6x - 6$$

$$B:\ f_{xy} = 6y$$

$$C:\ f_{yy} = 6x - 6$$

To find Stationary points:

$f_x = 0$

$\Rightarrow 3x^2+3y^2-6x = 0$

$\Rightarrow x^2+y^2-2x = 0$

$\Rightarrow y^2 = 2x - x^2$

$f_y = 0$

$6xy - 6y = 0$

$6y(x-1) = 0$

$\Rightarrow y = 0$ or $x-1=0$

$\Rightarrow y = 0$ or $x = 1$.

when $x = 1$;

$y^2 = 1$

$\Rightarrow y = \pm 1$

when $y = 0$

$\Rightarrow 0 = 2x - x^2$

$\Rightarrow x(2-x) = 0$

$\Rightarrow x = 0,\ x = 2$

$\therefore$ The stationary points are. $(0,0); (2,0), (1,1), (1,-1)$

| Stationary points | A $f_{xx}=6x-6$ | B $f_{xy}=6y$ | C $f_{yy}=6x-6$ | $AC-B^2$ | Conclusion |
|---|---|---|---|---|---|
| $(0,0)$ | $-6<0$ | $0$ | $-6$ | $36>0$ | Maximum point |
| $(2,0)$ | $6>0$ | $0$ | $6$ | $36>0$ | Minimum point |
| $(1,1)$ | $0$ | $6$ | $0$ | $-36<0$ | Saddle point |
| $(1,-1)$ | $0$ | $-6$ | $0$ | $-36<0$ | Saddle point |

To find maximum value:

Maximum value of $f(x,y)$ at the point $(0,0)$ is

$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$f(0,0) = 4$, a maximum value.

To find minimum value:

Minimum value of $f(x,y)$ at the point $(2,0)$ is

$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$f(2,0) = 8 + 0 - 12 - 0 + 4$

$= 0$, a minimum value.

② Find the max. & min. value of $f(x,y) = x^2 - xy + y^2 - 2x + y$

$f(x,y) = x^2 - xy + y^2 - 2x + y$.

$f_x = 2x - y - 2$.

$f_y = -x + 2y + 1$

A: $f_{xx} = 2$.

B: $f_{xy} = -1$

C: $f_{yy} = 2$

To find stationary points:

$f_x = 0 \Rightarrow 2x - y - 2 = 0$

$\Rightarrow 2x - 2 = y \quad \text{--- (1)}$

$f_y = 0 \Rightarrow -x + 2y + 1 = 0$

$\Rightarrow 2y = x - 1$

$\Rightarrow y = \frac{x-1}{2} \quad \text{--- (2)}$

From (1) & (2) $\Rightarrow 2x - 2 = \frac{x-1}{2}$

$\Rightarrow 4x - 4 = x - 1$

$\Rightarrow 3x = 3$

$\boxed{x = 1}$

when $x = 1$ in $2x - 2 = y$

$\Rightarrow \boxed{y = 0}$

∴ The stationary point is $(1, 0)$.

| Stationary point | A $f_{xx} = 2$ | B $f_{xy} = 1$ | C $f_{yy} = 2$ | $AC - B^2$ | Conclusion |
|---|---|---|---|---|---|
| $(1,0)$ | $2 > 0$ | $1$ | $2$ | $3 > 0$ | minimum point. |

To find minimum value:

$f(x,y) = x^2 - xy + y^2 - 2x + y$

$f(1,0) = 1 - 2 = -1$, a minimum value.

③ $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Soln: $f_x = 3x^2 - 3$

$f_y = 3y^2 - 12$

A: $f_{xx} = 6x$

C: $f_{yy} = 6y$

B: $f_{xy} = 0$

To find stationary points

$f_x = 0 \quad ; \quad f_y = 0$

$\Rightarrow 3x^2 - 3 = 0 \quad ; \quad 3y^2 - 12 = 0$

$\Rightarrow x = \pm 1 \quad ; \quad y = \pm 2$.

$\therefore$ The stationary pts are $(1,2), (1,-2), (-1,2), (-1,-2)$

| Stationary point | A $f_{xx} = 6x$ | B $f_{xy} = 0$ | C $f_{yy} = 6y$ | $AC - B^2$ | Conclusion |
|---|---|---|---|---|---|
| $(1,2)$ | $6 > 0$ | $0$ | $12$ | $72 > 0$ | minimum pt. |
| $(1,-2)$ | $6 > 0$ | $0$ | $-12$ | $-72 < 0$ | Saddle point |
| $(-1,2)$ | $-6 < 0$ | $0$ | $12$ | $-72 < 0$ | Saddle point |
| $(-1,-2)$ | $-6 < 0$ | $0$ | $-12$ | $72 > 0$ | Maximum point |

To find maximum value :

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f(-1,-2) = 38$, a max. value.

To find mini. value :

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f(1,2) = 2$, a mini. value.