Evaluate the stiffness matrix for the elements shown in Figure 0. The coordinates are given in units of millimeters. Assume plane stress conditions. Let E =210GPa, $\mu = 0.25$, and t = 10 mm.

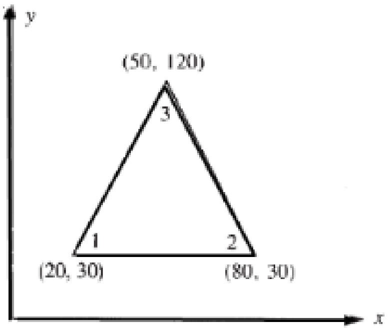
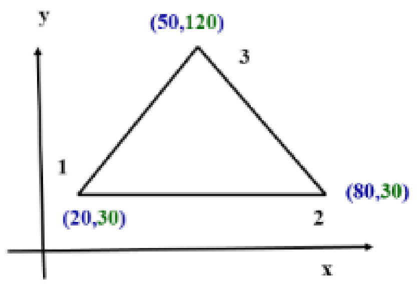


Figure 1.0

$$(x_1, y_1) = (20, 30)$$
- $$(x_2, y_2) = (80, 30)$$

$$(x_3, y_3) = (50, 120)$$

$$To\ find;\ [K] = [B]^T[D][B].A.t$$



Step i) Area of Triangular Element

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = A = \frac{1}{2}\begin{bmatrix} 1 & 20 & 30 \\ 1 & 80 & 30 \\ 1 & 50 & 120 \end{bmatrix}$$

$$= \frac{1}{2}(1(80 \times 120 - 30 \times 50) - 20(1 \times 120 - 30 \times 1) + 30(1 \times 50 - 80 \times 1))$$

$$A = 2700 mm^2$$

Step ii) Strain Displacement matrix

- $$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$q_1 = y_2 - y_3 = 30 - 120 = -90$ $\quad$ $r_1 = x_3 - x_2 = 50 - 80 = -30$

$q_2 = y_3 - y_1 = 120 - 30 = 90$ $\quad$ $r_2 = x_1 - x_3 = 20 - 50 = -30$

$q_3 = y_1 - y_2 = 30 - 30 = 0$ $\quad$ $r_3 = x_2 - x_1 = 80 - 20 = 60$

$$[B] = \frac{1}{2 \times 2700}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{2 \times 2700} \times [30] \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{180} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

•

$$[B]^T = \frac{1}{180} \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & 3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

ii) Stress − strain matrix [D] for plane stress condition: $[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$

$$[D] = \frac{2.1 \times 10^5}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$\mathrm{D} = \frac{2.1 \times 10^5}{1 - 0.25^2} [0.25] \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[\mathrm{D}] = 56000 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\bullet [K] = [B]^T [D][B].A.t$$

$$= [D][B]$$

$$= 56 \times 10^3 \times \frac{1}{180} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Matrix A[3x3] *Matrix B[3x3]* *Matrix C[3x3]*

$$= 311.111 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Matrix A[3x3]x Matrix B[3x3]

Matrix A[3x3]x Matrix C[3x3]

$$= 311.111 \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

$To\ find; [K] = [B]^T [D][B]. A. t$

- $= [B]^T [D][B].$

$$= \frac{1}{180} \times 311.111 \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

$$= 1.72839 \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

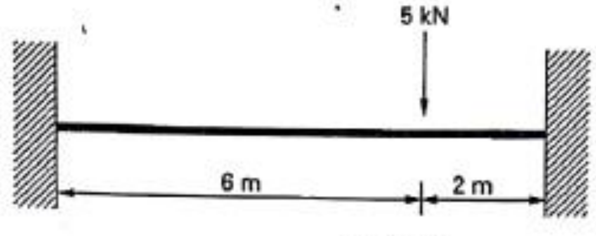
$$= 1.72839 \begin{array}{c} \textbf{mat } A^1 \\ \\ \textbf{mat } A^2 \end{array} \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{array}{cc} \textbf{mat } B & \textbf{mat } C \\ \multicolumn{2}{c}{\begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}} \end{array}$$

$$\begin{bmatrix} mat\ A^1 \times mat\ B & mat\ A^1 \times mat\ C \\ mat\ A^2 \times mat\ B & mat\ A^2 \times mat\ C \end{bmatrix}$$

$= A. t [B]^T [D][B]$

$$= 46666.53 \begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix}$$

Calculate the deflection under the load in the statically indeterminate beam in figure.1., and predict the shear force and bending moment distributions.

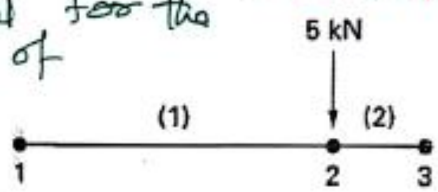


$\rightarrow 200\times10^{3}\times10^{6}\ N/m^2$

$1\,Pa = 10^{-6}\ N/mm^2$
$1\,MPa = 1\ N/mm^2$
$1\,ksi = 6.895\ N/mm^2$
$1\,bar = 0.1\ N/mm^2$

Figure .1.

The finite element model for the beam need only consist of two elements.



Stiffness matrix for Element (1)

$$K^{(1)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$= \frac{200\times10^{3}\times10^{6}\times4\times10^{-6}}{(6)^3}\begin{bmatrix} 12 & 6(6) & -12 & 6(6) \\ 6(6) & 4(6)^2 & -6(6) & 2(6)^2 \\ -12 & -6(6) & 12 & -6(6) \\ 6(6) & 2(6)^2 & -6(6) & 4(6)^2 \end{bmatrix}$$

$$= \frac{0.8\times10^{+6}}{216}\begin{bmatrix} 12 & 36 & -12 & 36 \\ 36 & 144 & -36 & 72 \\ -12 & -36 & 12 & -36 \\ 36 & 72 & -36 & 144 \end{bmatrix}$$

$$= 10^{-6}\begin{bmatrix} 0.044 & 0.133 & -0.044 & 0.133 \\ 0.133 & 0.533 & -0.133 & 0.266 \\ -0.044 & -0.133 & 0.044 & -0.133 \\ 0.133 & 0.266 & -0.133 & 0.533 \end{bmatrix}$$

Stiffness matrix for element (2)

$$= \frac{0.8\times10^{+6}}{2^3}\begin{bmatrix} 12 & 6(2) & -12 & 6(2) \\ 6(2) & 4(2)^2 & -6(2) & 2(2)^2 \\ -12 & -6(2) & 12 & -6(2) \\ 6(2) & 2(2)^2 & -6(2) & 4(2)^2 \end{bmatrix}$$

$$= 10^{+6}\times\left[\frac{1}{10}\right]\begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix}$$

$$= 10^{+6}\begin{bmatrix} 1.2 & 1.2 & -1.2 & 1.2 \\ 1.2 & 1.6 & -1.2 & 0.8 \\ -1.2 & -1.2 & +1.2 & -1.2 \\ 1.2 & 0.8 & -1.2 & 1.6 \end{bmatrix}$$

Including the load of 5kN applied at node 2, the final set of system equation is then $[K_G]\{U^T\} = \{F\}$

$$10^{+6}\begin{array}{cc} & \begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ & \begin{bmatrix} 0.044 & 0.133 & -0.044 & 0.133 & 0.0 & 0.0 \\ 0.133 & 0.533 & -0.133 & 0.266 & 0.0 & 0.0 \\ -0.044 & -0.133 & 1.244 & 1.067 & -1.2 & 1.2 \\ 0.133 & 0.266 & 1.067 & 2.133 & -1.2 & 0.8 \\ 0.0 & 0.0 & -1.2 & -1.2 & 1.2 & -1.2 \\ 0.0 & 0.6 & 1.2 & 0.8 & -1.2 & 0.8 \; 1.6 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$\times \begin{bmatrix} \cancel{v_1} \to 0 \\ \cancel{\theta_1} \to 0 \\ v_2 \\ \theta_2 \\ \cancel{v_3} \to 0 \\ \cancel{\theta_3} \to 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -5\times10^3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

However, before the equation can be solved, the constant conditions must be applied, namely $v_1 = \theta_1 = v_3 = \theta_3 = 0$

These result in reduced set of equation

$$10^{+6} \times \begin{bmatrix} 1.244 & 1.067 \\ 1.067 & 2.133 \end{bmatrix} \begin{bmatrix} v_2 \\ \theta_2 \end{bmatrix} = \begin{Bmatrix} -5\times10^3 \\ 0 \end{Bmatrix}$$

$$1.244\times10^{+6} v_2 + 1.067\times10^{+6}\theta_2 = -5\times10^3 \quad \text{— ①}$$

$$1.067\times10^{+6} v_2 + 2.133\times10^{+6}\theta_2 = 0 \quad \text{— ②}$$

Solve

$v_2 = -0.007037\ m \qquad \theta_2 = +0.00352\ rad$

To illustrate how we can combine spring and bar elements in one structure, we now solve the two-bar truss supported by a spring shown in Figure .1. Both bars have E= 210 GPa and $A=5\times10^4 m^2$. Bar one has a length of 5 m and bar two a length of 10 m. The spring stiffness is k = 2000 kN/m.

$l = \cos\theta = \frac{x_2 - x_1}{l_e}$

$m = \sin\theta = \frac{y_2 - y_1}{l_e}$

$l_e = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

Two-bar truss with spring support

Solution:

Stiffness matrix

$$K = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Element ① $\theta = 135°$

$l = \cos 135 = -707.10678\times10^{-3}$

$l^2 = 0.5$

$m = \sin 135 = +707.10678\times10^{-3}$

$m^2 = 0.5$ $[lm = -0.5]$

Stiffness matrix for Element ①

$L = 5m$ [node (1-2)]

$$K^{(1)} = \frac{(5\times10^{-4})\times(210\times10^{9})}{5}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$= 105\times10^{5} \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Element ②, $\theta = 180^\circ$

$l^2 = 1 \quad lm = 0 \quad m^2 = 0$

Stiffness matrix for Element ② node [1–3]

$L = 10\,m$

$$K^{(2)} = \frac{(5\times10^{-4})\times(210\times10^{9})}{10}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 105\times10^{5}\ \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

Element ③, $\theta = 270^\circ$

$l^2 = 0, \quad lm = 0 \quad m^2 = 1$

Stiffness matrix for Element ③ node [1–4] $= K[\ ]$

$$K^{(3)} = 20\times10^{5}\ \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \end{array}$$

General Finite Element equation: $[K]\{u\}=\{F\}$

$$
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}\\
\begin{bmatrix}
105+105 & -105 & -105 & 105 & -105 & 0 & 0 & 0 \\
-105 & 105+20 & 105 & -105 & 0 & 0 & 0 & -20 \\
-105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 \\
105 & -105 & -105 & 105 & 0 & 0 & 0 & 0 \\
-105 & 0 & 0 & 0 & +105 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -20 & 0 & 0 & 0 & 0 & 0 & 20
\end{bmatrix}
\end{array}
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8 \end{array}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \to 0 \\ u_4 \to 0 \\ u_5 \to 0 \\ u_6 \to 0 \\ u_7 \to 0 \\ u_8 \to 0 \end{bmatrix}
=
\begin{bmatrix} F_1(0) \\ F_2(-50) \\ F_3(0) \\ F_4(0) \\ F_5(0) \\ F_6(0) \\ F_7(0) \\ F_8(0) \end{bmatrix} \times 10^3
$$

Boundary conditions are

$u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$

The final matrix

$$10^5 \begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50 \times 10^3 \end{Bmatrix}$$

$$210\times10^5 u_1 - 105\times10^5 u_2 = 0 \quad -(1)$$
$$-105\times10^5 u_1 + 125\times10^5 u_2 = -50\times10^3 \quad -(2)$$

$u_1 = -3.448\times10^{-3}\,m$

$u_2 = -6.896\times10^{-3}\,m$

Stress : Stress of element (1)

$$\sigma^{(1)} = \frac{E}{L_{(1)}}\begin{bmatrix}-l_1 & -m_1 & l_1 & m_1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}$$

$$\sigma^{(1)}_{(1-2)} = \frac{210\times10^9}{5}\begin{bmatrix}0.707 & -0.707 & -0.707 & 0.707\end{bmatrix}\begin{Bmatrix}-3.448\times10^{-3}\\-6.896\times10^{-3}\\0\\0\end{Bmatrix}$$

$$= 102.4\,MPa\,[T]$$

$$\sigma^{(2)}_{(1-3)} = \frac{E}{L_2}\begin{bmatrix}-l_2 & -m_2 & l_2 & m_2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_5\\u_6\end{bmatrix}$$

$$= \frac{210\times10^9}{10}\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}-3.448\times10^{-3}\\-6.896\times10^{-3}\\0\\0\end{bmatrix}$$

$$= -72\,MPa\,[C]$$

$\sigma = P/A$

$f_{1-2} = 102.4 \times 5\times10^{-4}$ (MPa $\times10^6$)
$= 51.2\,kN$

$f_{1-3} = -72.4 \times 5\times10^{-4}$ (MPa $\times10^6$)
$= 35.6\,kN$

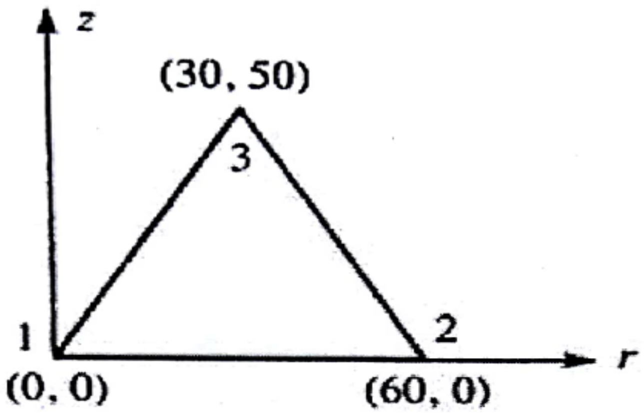
Note : can show equilibrium at node 1

$$F_s = [2000\,kN/m] \times [6.896\times10^{-3}\,m] = 13.792\,kN$$

$f_{1-3} = 35.6\,kN$
$f_{1-2} = 51.2\,kN$

$\sum F_y = 0$
$-50 + 13.79 + 36.198 = 0$

For the axisymmetric elements shown in Figure, determine the element stresses. Let $E = 210GPa$ and $\mu = 0.25$.The coordinates (in millimeters) are shown in the figures, and the nodal displacements for the element are $u_1 = 0.05mm, u_2 = 0.02mm, u_3 = 0.0mm$, $w_1 = 0.03mm, w_2 = 0.02mm, w_3 = 0.0mm$.



Given:

Coordinates :

$r_1 = 0$ mm, $z_1 = 0$ mm

$r_2 = 60$ mm, $z_2 = 0$ mm

$r_3 = 30$ mm. $z_3 = 50$ mm

Nodal displacements:

$u_1 = 0.05$ mm, $w_1 = 0.03$ mm

$u_2 = 0.02$ mm, $w_2 = 0.02$ mm

$u_3 = 0$ mm $w_3 = 0$ mm.

Youngs modulus, $E = 210$ GPa

Poisson's ratio, $\mu = 0.25$.

To find :

Element stresses :

i) Radial stress, σ_r

ii) Circumferential stress, σ_θ

iii) Longitudinal stress, σ_z

iv) shear stress, τ_{rz}.

Solution :

Stress $\{\sigma\} = [D][B][u]$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = [D][B] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \rightarrow ①$$

Area of the triangular element, $A = \frac{1}{2} \times B \times H$

$$= \frac{1}{2} \times 60 \times 50$$

$$= 1500 \text{ mm}^2 \quad \rightarrow ②$$

$$A = 1500 \text{ mm}^2.$$

coordinate, $r = \dfrac{r_1 + r_2 + r_3}{3} = \dfrac{0 + 60 + 30}{3}$

$$r = 30 \text{ mm} \quad \rightarrow ③$$

Coordinate, $z = \dfrac{z_1 + z_2 + z_3}{3} = \dfrac{0 + 0 + 50}{3}$

$$z = 16.667 \text{ mm} \quad \rightarrow ④$$

Stress - strain relationship matrix, [D]

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$[D] = \frac{2.1\times10^5}{(1+0.25)(1-(2\times0.25))}\begin{bmatrix}1-0.25 & 0.25 & 0.25 & 0\\ 0.25 & 1-0.25 & 0.25 & 0\\ 0.25 & 0.25 & 1-0.25 & 0\\ 0 & 0 & 0 & \frac{1-(2\times0.25)}{2}\end{bmatrix}$$

$$= 336\times10^3\begin{bmatrix}0.75 & 0.25 & 0.25 & 0\\ 0.25 & 0.75 & 0.25 & 0\\ 0.25 & 0.25 & 0.75 & 0\\ 0 & 0 & 0 & 0.25\end{bmatrix} = 336\times10^3\times0.25\begin{bmatrix}3 & 1 & 1 & 0\\ 1 & 3 & 1 & 0\\ 1 & 1 & 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[D] = 84\times10^3\begin{bmatrix}3 & 1 & 1 & 0\\ 1 & 3 & 1 & 0\\ 1 & 1 & 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \rightarrow ⑤$$

Strain - Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix}\beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0\\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0\\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3\\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3\end{bmatrix} \rightarrow ⑥$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = (60\times50) - (30\times0)$

$\alpha_1 = 3000\ mm^2$

$\alpha_2 = r_3 z_1 - r_1 z_3 = (30\times0) - (0\times50)$

$\alpha_2 = 0$

$\alpha_3 = r_1 z_2 - r_2 z_1 = (0\times0) - (60\times0)$

$\alpha_3 = 0$

$\beta_1 = z_2 - z_3 = 0 - 50$

$\beta_1 = -50\ mm$

$\beta_2 = z_3 - z_1 = 50 - 0$

$\beta_2 = 50\ mm.$

$\beta_3 = z_1 - z_2 = 0 - 0.$

$\beta_3 = 0$

$\gamma_1 = r_3 - r_2 = 30 - 60$

$\gamma_1 = -30$ mm.

$\gamma_2 = r_1 - r_3 = 0 - 30$

$\gamma_2 = -30$ mm

$\gamma_3 = r_2 - r_1 = 60 - 0$

$\gamma_3 = 60$ mm.

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{3000}{30} + (-50) + \frac{(-30 \times 16.667)}{30}$$

$$= 33.33 \text{ mm}$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = 0 + 50 + \frac{(-30 \times 16.667)}{30}$$

$$= 33.33 \text{ mm}$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = 0 + 0 + \frac{(60 \times 16.667)}{30}$$

$$= 33.33 \text{ mm}$$

Substitute values in equation ⑥

$$[B] = \frac{1}{2 \times 1500} \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$[B] = 3.3333 \times 10^{-4} \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 83.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix} \rightarrow (7)$$

$$[D][B] = 84 \times 10^{3} \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times 3.3333 \times 10^{-4} \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$= 28 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$[D][B] = 28 \begin{bmatrix} -116.67 & -30 & 183.33 & -30 & 33.33 & 60 \\ 49.99 & -30 & 149.99 & -30 & 99.99 & 60 \\ -16.67 & -90 & 83.33 & -90 & 33.33 & 180 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix} \rightarrow (8)$$

Substitute the values in equation (1).

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau \phi_{rz} \end{Bmatrix} = 28 \begin{bmatrix} -116.67 & -30 & 183.33 & -30 & 33.33 & 60 \\ 49.99 & -30 & 149.99 & -30 & 99.90 & 60 \\ -16.67 & -90 & 83.33 & -90 & 33.33 & 180 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = 28 \begin{bmatrix} -3.666 \\ 4 \\ -3.666 \\ -2.6 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \begin{Bmatrix} -103 \\ 112 \\ -103 \\ -73 \end{Bmatrix}$$

Result :

Element stresses :

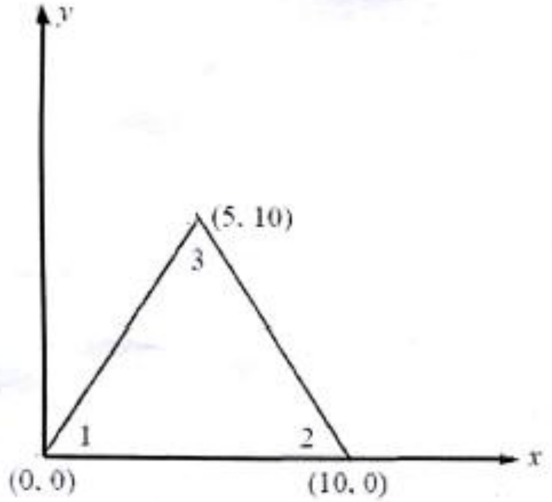
Radial stress, $\sigma_r = -103\ N/mm^2$

Circumferential stress, $\sigma_\theta = 112\ N/mm^2$

Longitudinal stress, $\sigma_z = -1.03\ N/mm^2$

Shear stress, $\tau_{rz} = -73\ N/mm^2$

Evaluate the stiffness matrix for the elements shown in Figure 0. The coordinates are given in units of millimeters. Assume plane stress conditions. Let E =210GPa, $\mu = 0.25$, and t = 10 mm.



$(x_1, y_1) = (0,0)$

$(x_2, y_2) = (10,0)$

$(x_3, y_3) = (5,10)$

To Find:-

$$[k] = [B]^T [D][B] \cdot A \cdot t$$

Step i) Area of triangular Element.

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = A = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 5 & 10 \end{bmatrix}$$

$$= \frac{1}{2}\left[1(10\times10 - 0) - 0 + 0\right]$$

$$A = \frac{100}{2} = 50\,mm^2$$

Step ii) Strain Displacement matrix

$$B = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$\therefore q_1 = y_2 - y_3 = 0 - 10 = -10$

$q_2 = y_3 - y_1 = 10 - 0 = 10$

$q_3 = y_1 - y_2 = 0 - 0 = 0$

$r_1 = x_3 - x_2 = 5 - 10 = -5$

$r_2 = x_1 - x_3 = 0 - 5 = -5$

$r_3 = x_2 - x_1 = 10 - 0 = 10$

1/8

$$[B] = \frac{1}{2 \times 50} \times 5 \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -2 & -1 & 2 & 2 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{20} \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -2 & -1 & 2 & 2 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{20} \begin{bmatrix} -2 & 0 & -1 \\ 0 & -1 & -2 \\ 2 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Stress strain matrix [D] for plane stress condition:

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$[D] = \frac{2.1 \times 10^5}{1-0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D] = \frac{2.1 \times 10^5}{1-0.25^2} [0.25] \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

2/4

$$[D] = 56000 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[k] = [B]^T [D] [B] \cdot A \cdot t$$

$$= [D][B]$$

$$= 56 \times 10^3 \times \frac{1}{20} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times$$

$$\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -2 & -1 & 2 & 2 & 0 \end{bmatrix}$$

$$= 2800 \begin{bmatrix} -8 & -1 & 8 & -1 & 0 & 2 \\ -2 & -4 & 2 & -4 & 0 & 8 \\ -1.5 & -3 & -1.5 & 3 & 3 & 0 \end{bmatrix}$$

To find:

$$[k] = [B]^T [D] [B] \cdot A \cdot t$$

$$= [B]^T [D] [B]$$

$$= \frac{2800}{2} \begin{bmatrix} -2 & 0 & -1 \\ 0 & -1 & -2 \\ 2 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -8 & -1 & 8 & -1 & 0 & 2 \\ -2 & -4 & 2 & -4 & 0 & 8 \\ -1.5 & -3 & -1.5 & 3 & 3 & 0 \end{bmatrix}$$

$$
= 70000 \begin{bmatrix} 17.5 & 5 & -14.5 & -1 & -3 & -4 \\ 5 & 10 & 1 & -2 & -6 & -8 \\ -14.5 & 1 & 17.5 & -5 & -3 & 4 \\ -1 & -2 & -5 & 10 & 6 & -8 \\ -3 & -6 & -3 & 6 & 6 & 0 \\ -4 & -8 & 4 & -8 & 0 & 16 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1.225\times10^{6} & 3.5\times10^{5} & -1.015\times10^{6} & -7\times10^{4} & -2.1\times10^{5} & -2.8\times10^{5} \\ 3.5\times10^{5} & 7\times10^{5} & 7\times10^{4} & -1.4\times10^{5} & -4.2\times10^{5} & -5.6\times10^{5} \\ -1.015\times10^{6} & 7\times10^{4} & 1.225\times10^{6} & -3.5\times10^{5} & -2.1\times10^{5} & 2.8\times10^{5} \\ -7\times10^{4} & -1.4\times10^{5} & -3.5\times10^{5} & 7\times10^{5} & -4.2\times10^{5} & -5.6\times10^{5} \\ -2.1\times10^{5} & -4.2\times10^{5} & -2.1\times10^{5} & 4.2\times10^{5} & -4.2\times10^{5} & 0 \\ -2.8\times10^{5} & -5.6\times10^{5} & 2.8\times10^{5} & -5.6\times10^{5} & 0 & 1.12\times10^{6} \end{bmatrix}
$$

Temperature effects: When the change in temperature ΔT is present, then strain ε_o should be considered,

For plane stress $\{\varepsilon_o\} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ O \end{Bmatrix}$ For Plane strain : $\{\varepsilon_0\} = (1+\mu)\begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ O \end{Bmatrix}$

$\{\sigma\} = [D].\{\varepsilon - \varepsilon_0\}$ α - Coeff of the expression

$\{\sigma\} = [D].\ \{[B]\{\mu\} - \{\varepsilon_o\}\ \}$

The element temp force can be represented by

$\{F\} = [B]^T [D]\{\varepsilon_0\} t.A$

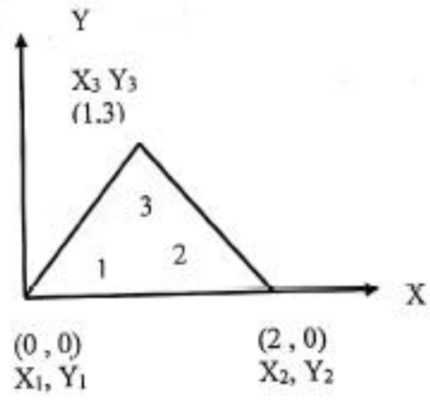
Calculate element stiffness matrix and temperature force vector for the plane stress element shown in figure. The element expressions 20 degree increase in temperature. Assume $\alpha = 6\times10^{-6} /^0 C, \varepsilon = 2\times10^5 N/mm^2.\ \mu = 0.25, t = 5mm.$

Given: $\alpha = 6\times10^{-6} /^0 C; \varepsilon = 2\times10^5 N/mm^2; \mu = 0.25; t = 5mm$

$\Delta T = 20^0 C$, To find $[K]\ \{F\}$

WKT, $[K] = [B]^T [D].[B] t.A$

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$



$$Aera[A] = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{vmatrix}$$

= (6 -0)

2A = 6

A=3

$q_1 = y_2 - y_3 = 0 - 3 = -3$

$q_2 = y_3 - y_1 = 3 - 0 = 3$

$q_3 = y_1 - y_2 = 0 - 0 = 0$

$r_1 = x_2 - x_3 = 1 - 2 = -1$

$r_2 = x_1 - x_3 = 0 - 1 = -1$

$r_3 = x_2 - x_1 = 2 - 0 = 2$

$$[D]=\frac{E}{1-\mu^2}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \frac{1-\mu}{2}\end{bmatrix}$$

$$=\frac{2\times10^5}{1-(0.25)^2}\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & \frac{1-0.25}{2}\end{bmatrix}$$

$$=213.33\times10^3\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}$$

$$[B]=\frac{1}{6}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\ 0 & -1 & 0 & -1 & 0 & 2\\ -1 & -3 & -1 & 3 & 2 & 0\end{bmatrix}$$

$$[B]^T=\frac{1}{6}\begin{bmatrix}-3 & 0 & -1\\ 0 & -1 & -3\\ 3 & 0 & -1\\ 0 & -1 & 3\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}$$

$$[B]^T[D]=\frac{1}{6}\begin{bmatrix}-3 & 0 & -1\\ 0 & -1 & -3\\ 3 & 0 & -1\\ 0 & -1 & 3\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}\times213.33\times10^3\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}$$

$$=0.35555\times10^5\begin{bmatrix}-3 & -0.75 & -0.375\\ -0.25 & -1 & -1.125\\ 3 & 0.75 & -0.375\\ -0.25 & -1 & 1.125\\ 0 & 0 & 0.75\\ 0.5 & 2 & 0\end{bmatrix}$$

$$[K] = 0.35555\times10^5 \begin{bmatrix} -3 & -0.75 & -0.375 \\ -0.25 & -1 & -1.125 \\ 3 & 0.75 & -0.375 \\ -0.25 & -1 & 1.125 \\ 0 & 0 & 0.75 \\ 0.5 & 2 & 0 \end{bmatrix} \times \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix} \times 5 \times 3$$

$$= 88750 \begin{bmatrix} 9.375 & 1.875 & -8.625 & -0.375 & -0.75 & -1.5 \\ 1.875 & 4.375 & 0.375 & -2.375 & -2.25 & -2 \\ -8.625 & 0.375 & 9.375 & -1.875 & -0.75 & 1.5 \\ -0.375 & -2.375 & -1.875 & 4.375 & 2.25 & -2 \\ -0.75 & -2.25 & -0.75 & 2.25 & 1.5 & 0 \\ -1.5 & -2 & 1.5 & -2 & 0 & 1 \end{bmatrix}$$

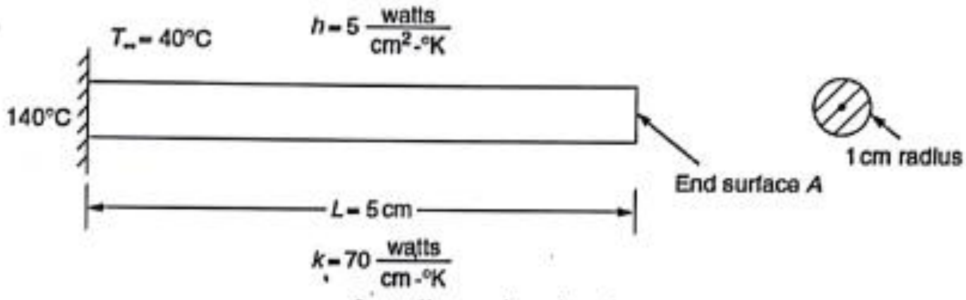
$$\{\varepsilon_0\} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6\times10^{-6}\times20 \\ 6\times10^{-6}\times20 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.2\times10^{-4} \\ 1.2\times10^{-4} \\ 0 \end{Bmatrix}$$

$$\{F\} = [\beta]^T [D]\{C_0\} t.A$$

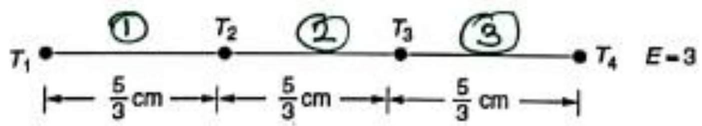
$$0.355\times10^5 \begin{bmatrix} -3 & -0.75 & -0.375 \\ -0.25 & -1 & -1.125 \\ 3 & 0.75 & -0.375 \\ -0.25 & -1 & 1.125 \\ 0 & 0 & 0.75 \\ 0.5 & 2 & 0 \end{bmatrix} \times 5 \times 3 \times \begin{Bmatrix} 1.2\times10^{1} \\ 1.2\times10^{1} \\ 0 \end{Bmatrix} \qquad 6 \text{ X } 3,\ 3 \text{ X } 1$$

$$\{F\} = 5.325\times10^5 \begin{Bmatrix} 2.17\times10^{-4} \\ -1.5\times10^{-4} \\ 4.5\times10^{-4} \\ -1.5\times10^{-4} \\ 0 \\ 3\times10^{-4} \end{Bmatrix} N$$

A fin is a one-dimensional heat transfer problem. One end of the fin is connected to a heat source (whose temperature is known $140°C$) and heat will be lost to the surroundings through the perimeter surface and the end. The temperature of the ambient air is $40°C$. The thermal conductivity of is $70(\frac{watts}{cm°K})$. The natural convective heat transfer coefficient associated with the surrounding air is $5\frac{watts}{cm^2 °K}$. Find the temperature distribution in the fin shown in Figure 1.0 by including the effect of convection from the end surface A using three finite elements.



One dimensional rod



Finite element idealization

For elements (1) and (2) in the situation, the conductance and thermal load matrix are given by

$$[K]^{(e)} = \left\{ \frac{kA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hpl}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right]$$

From axial conduction perimeter convection.

$$[F]^{(e)} = \frac{hpl T_f}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ perimeter convection.

Substituting for properties, we obtain.

$$[K]^{(1)} = \left\{ \frac{70 \times \pi (1)^2}{5/3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{5 \times 2\pi \times 1 \times 5/3}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 132.05 & -132.05 \\ -132.05 & 132.05 \end{bmatrix} + \begin{bmatrix} 17.46702 & 8.733508 \\ 8.733508 & 17.46702 \end{bmatrix}$$

$$= \begin{array}{c c} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 149.502 & -123.3165 \\ -123.3165 & 149.502 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$[K]^{(1)} = [K]^{(2)} = \begin{array}{c c} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 149.502 & -123.3165 \\ -123.3165 & 149.502 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

The thermal-load matrix for element 1, 2

$$F^{(1)} = F^{(2)} = \frac{5 \times 2\pi \times 1 \times 5/3 \times 40}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{array}{cc} \begin{bmatrix} 1048.021 \\ 1048.021 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$F^{(2)} = \begin{array}{cc} \begin{bmatrix} 1048.021 \\ 1048.021 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Including the boundary condition of the tip the conductance and load matrix for element (3) are obtain in the following manner.

$$K^{(e)} = \left\{ \underbrace{\frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{From axial conduction}} + \underbrace{\frac{hpl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{perimeter convection}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}}_{\text{end convection at node j}} \right\}$$

$$= \begin{bmatrix} 149.502 & -123.3165 \\ -123.3165 & 149.502 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 5\times\pi(1)^2 \end{bmatrix}$$

$$= \begin{array}{cc} 3 & 4 \\ \left[\begin{matrix} 149.502 & -123.3165 \\ -123.3165 & 165.222 \end{matrix}\right] & \begin{matrix} 3 \\ 4 \end{matrix} \end{array}$$

load matrix F^3

$$= \frac{hplT_f}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ hAT_f \end{bmatrix}$$

$$= \begin{bmatrix} 1048.21 \\ 1048.21 \end{bmatrix} + \begin{bmatrix} 0 \\ 628.8 \end{bmatrix}$$

$$= \begin{bmatrix} 1048.021 \\ 1676.821 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

Assembly of the elements conductance [K] and global loads to the global load matrix (F)

$[K][T] = [F]$

$$\begin{bmatrix} 149.502 & -123.3165 & 0 & 0 \\ -123.3165 & \boxed{149.502 + 149.502} & -123.3165 & 0 \\ 0 & -123.3165 & \boxed{149.502 + 149.502} & -123.3165 \\ 0 & 0 & -123.3165 & 165.222 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 1048.02 \\ 2096.04 \\ 2096.042 \\ 1676.821 \end{bmatrix}$$

After incorporating the boundary condition $T_1 = 140°C$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -123.3165 & 299.004 & -123.3165 & 0 \\ 0 & -123.3165 & 299.004 & -123.3165 \\ 0 & 0 & -123.3165 & 165.222 \end{bmatrix} \begin{bmatrix} T_1 \xrightarrow{} 140°C \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2096.042 \\ 2096.042 \\ 1676.821 \end{bmatrix}$$

$-123.3165 \underset{}{[T_1]}^{140°c} + 299.004 [T_2] - 123.3165 [T_3] = 2096.042$

$299.004 T_2 - 123.3165 T_3 = 2096.042 + (123.3165 \times 140)$

$= 19360.352 \quad \text{—} (1)$

$-123.3165 T_2 + 299.004 T_3 - 123.3165 T_4 = 2096.042 \quad \text{—} (2)$

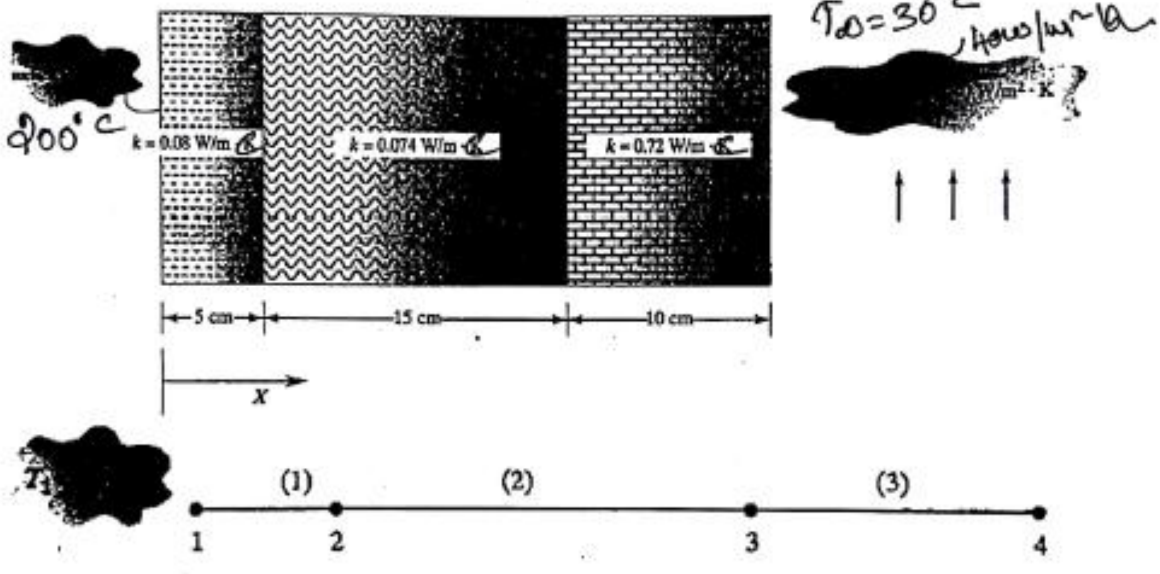
$-123.3165 T_3 + 165.222 T_4 = 1676.82 \quad \text{—} (3)$

Solve above equation.

$T_1 = 140°$ $\quad T_2 = 94.68876°C$ $\quad T_3 = 72.5934°C$

$T_4 = 64.3303°C$

A wall of an industrial oven consists of three different materials, as depicted in figure 1.0. The first layer is composed of 5cm of insulating cement with a clay binder that has a thermal conductivity of 0.08W/m.K. The second layer is made from 15cm of 6-ply asbestos board with a thermal conductivity of 0.074W/m.K. The exterior consists of 10cm common brick with a thermal conductivity of $0.72 W/m^2 K$. The inside wall temperature of oven is 200 $°C$, and the outside air is $30°C$ with a convection coefficient of $40W/m^2.K$. Determine the temperature distribution along the composite wall.



A composite wall of an industrial oven.

Stiffness matrix for Element 1

$$K^{[1]} = \frac{KA}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0.08\times 1}{0.05}\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix} = \begin{array}{cc} 1 & 2 \\ \left[\begin{matrix}1.6 & -1.6\\ -1.6 & 1.6\end{matrix}\right] & \begin{matrix}1\\2\end{matrix}\end{array} W/°C$$

Stiffness matrix for Element 2

$$K^{(2)} = \frac{K_2 A}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0.074\times 1}{0.15}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{array}{cc} 2 & 3 \\ \left[\begin{matrix}0.493 & -0.493\\ -0.493 & 0.493\end{matrix}\right] & \begin{matrix}2\\3\end{matrix}\end{array} W/°C$$

For Element (3), including the boundary conditions.

$$[K]^{(3)} = \frac{k_3 A}{L_3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & hA\end{bmatrix}$$

$$= \frac{0.72\times 1}{0.1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 40\times 1\end{bmatrix}$$

$$= \begin{array}{cc} \begin{array}{cc} 3 & \quad 4 \end{array} & \\ \begin{bmatrix}7.2 & -7.2\\ -7.2 & 47.2\end{bmatrix} & \begin{array}{c} 3\\ 4 \end{array} \end{array}$$

Thermal load / force Vector.

$$\{F\}^{(1)} = \begin{Bmatrix}0\\0\end{Bmatrix} W \qquad F^{(2)} = \begin{Bmatrix}0\\0\end{Bmatrix} W$$

$$\{F\}^{(3)} = \begin{Bmatrix}0\\ hAT_f\end{Bmatrix} = \begin{Bmatrix}0\\ 40\times 1\times 30\end{Bmatrix} = \begin{bmatrix}0\\1200\end{bmatrix} W$$

Assembling elements, we obtain

$$[K]^{(G)} = \begin{bmatrix}1.6 & -1.6 & 0 & 0\\ -1.6 & 1.6+0.493 & -0.493 & 0\\ 0 & -0.493 & 0.493+7.2 & -7.2\\ 0 & 0 & -7.2 & 47.2\end{bmatrix}$$

$$[F]^{(G)} = \begin{bmatrix}0\\0\\0\\1200\end{bmatrix}$$

Apply the boundary condition at the inside furnace wall, we get.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.6 & 2.093 & -0.493 & 0 \\ 0 & -0.493 & 7.693 & -7.2 \\ 0 & 0 & -7.2 & 47.2 \end{bmatrix} \begin{Bmatrix} T_1 \xrightarrow{} 200 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1200 \end{Bmatrix}$$

and solving the set of linear equations, we have the following results.

$$-1.6 \times T_1^{\to 200} + 2.093 T_2 - 0.493 T_3 = 0$$

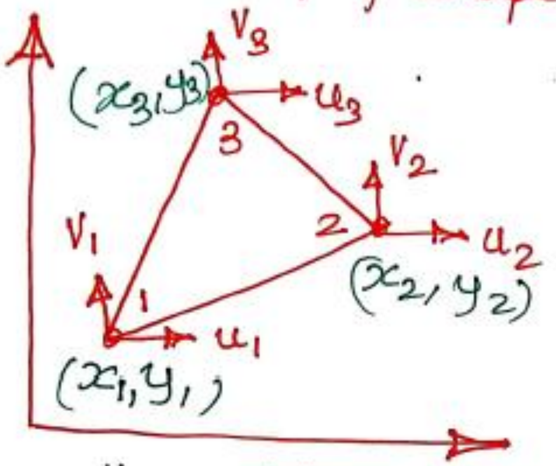
$$2.093 T_2 - 0.493 T_3 = 320 \quad \text{---} (1)$$

$$-0.493 T_2 + 7.693 T_3 - 7.2 T_4 = 0 \quad \text{---} (2)$$

$$-7.2 T_3 + 47.2 T_4 = 1200 \quad \text{---} (3)$$

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 162.3 \\ 39.9 \\ 31.5 \end{Bmatrix} {}^\circ C$$

Derivation of shape function for CST



Let the nodal displacements be

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

u_1, u_2, u_3 → displacement along x axis.

v_1, v_2, v_3 → displacement along y axis

Since a CST has 3 nodes and each node has 2 dof, totally 6 dof. Hence it should be approximated with 6 generalized co-ordinates. (a_1 to a_6).

$$u = a_1 + a_2 x + a_3 y$$

$$v = a_4 + a_5 x + a_6 y$$

Now, apply the nodal conditions.

$u_1 = a_1 + a_2 x_1 + a_3 y_1 \quad v_1 = a_4 + a_5 x_1 + a_6 y_1$

$u_2 = a_1 + a_2 x_2 + a_3 y_2 \quad v_2 = a_4 + a_5 x_2 + a_6 y_2$

$u_3 = a_1 + a_2 x_3 + a_3 y_3 \quad v_3 = a_4 + a_5 x_3 + a_6 y_3$

In matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$x = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$x^{-1} = \frac{C^T}{|x|}$$

$$x^{-1} = \frac{\begin{bmatrix} (x_2 y_3 - x_3 y_2) & -(y_3 - y_2) & (x_3 - x_2) \\ -(x_1 y_3 - x_3 y_1) & (y_3 - y_1) & -(x_3 - x_1) \\ (x_1 y_2 - x_2 y_1) & -(y_2 - y_1) & (x_2 - x_1) \end{bmatrix}^T}{(x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)} \Rightarrow 2A$$

2D- problem

$$x^{-1} = \frac{1}{2A}\begin{bmatrix}(x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1)\\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2)\\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1)\end{bmatrix}$$

$$\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix} = \frac{1}{2A}\begin{bmatrix}(x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1)\\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2)\\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1)\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

$$U = [1 \quad x \quad y]\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = [1 \quad x \quad y]\frac{1}{2A}\begin{bmatrix}p_1 & p_2 & p_3\\ q_1 & q_2 & q_3\\ r_1 & r_2 & r_3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{p_1 + q_1x + r_1y}{2A} & \frac{p_2 + q_2x + r_2y}{2A} & \frac{p_3 + q_3x + r_3y}{2A}\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

This is form of.

$$u = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix} \qquad \text{||ly} \quad V = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix}v_1\\v_2\\v_3\end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix}u\\v\end{Bmatrix} = \begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3\end{bmatrix}\begin{Bmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\end{Bmatrix}$$

where

$$N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}$$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

$p_1 = x_2 y_3 - x_3 y_2 \quad q_1 = y_2 - y_3 \quad r_1 = x_3 - x_2$

$p_2 = x_3 y_1 - x_1 y_3 \quad q_2 = y_3 - y_1 \quad r_2 = x_1 - x_3$

$p_3 = x_1 y_2 - x_2 y_1 \quad q_3 = y_1 y_2 \quad r_3 = x_2 - x_1$

unit-3 2D problem

Strain - Displacement [B] matrix [Gradient matrix]

HKT

Strain in x-direction

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} \cdot u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$$

Strain in y-direction

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} \cdot v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} \cdot v_3$$

Strain along xy direction.

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y} \cdot u_1 + \frac{\partial N_2}{\partial y} \cdot u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} \cdot v_1 + \frac{\partial N_2}{\partial x} \cdot v_2 + \frac{\partial N_3}{\partial x} \cdot v_3$$

In matrix form

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_2}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_2}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}; \quad N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}; \quad N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

$$\frac{\partial N_1}{\partial x} = \frac{q_1}{2A}; \quad \frac{\partial N_2}{\partial x} = \frac{q_2}{2A}; \quad \frac{\partial N_3}{\partial x} = q_3/2A$$

$$\frac{\partial N_1}{\partial y} = r_1/2A; \quad \frac{\partial N_2}{\partial y} = \frac{r_2}{2A}; \quad \frac{\partial N_3}{\partial y} = r_3/2A$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

This is of the form

$$\{\varepsilon\} = [B]\{u\}$$

Stress - Displacement Relation

$$\{\sigma\} = [D] \cdot \{\varepsilon\}$$

$$\{\sigma\} = [D] \cdot [B] \cdot \{u\}$$

Max. Normal Stress

$$\sigma_1 = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

$$\sigma_2 = \frac{1}{2}\left[(\sigma_x + \sigma_y) - \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

principle angle $\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$

WKT stiffness matrix [K] for CST

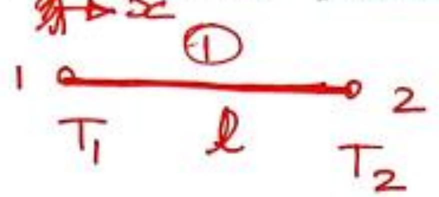
$$[K] = \int_v [B]^T [D][B] \cdot dv$$

$$= [B]^T [D][B] \cdot A \cdot t$$

where t = Thickness of body in case of plane stress

= 1 in case of plane strain

Derivation of stiffness matrix for one dimensional heat conduction element.



one dimensional bar element
T_1 & T_2 temp at the respective node 1,2
k - thermal conductivity of the material.

Stiffness matrix $[K] = \int [B]^T [D][B]\, dx$

Temperature function $T = N_1 T_1 + N_2 T_2$

$$N_1 = \frac{l-x}{l} \qquad N_2 = x/l$$

Strain-Displacement matrix $[B] = \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx}\right]$

$$[B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \qquad [B]^T = \begin{bmatrix} -1/l \\ 1/l \end{bmatrix}$$

In one dimensional heat conduction problem.

$[D] = [k] = k$ = Thermal conductivity of the material

Substitute $[B]$, $[B]^T$ and $[D]$ values in stiffness matrix equation.

$dv = A \cdot dx$

$\Rightarrow$ Stiffness matrix for heat conduction $\}$ $[K_c] = \int_0^l \begin{Bmatrix} -1/l \\ 1/l \end{Bmatrix} \times k \times [-1/l \;\; 1/l]\, dv$

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} k\, A \cdot dx$$

$$= Ak \begin{bmatrix} \frac{1}{l^2} & -\frac{1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^l dx$$

$$= Ak \begin{bmatrix} \frac{1}{l^2} & -\frac{1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$= Ak \begin{bmatrix} \frac{1}{l^2} & -\frac{1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix} [l-0]$$

$$= \frac{Ak\cancel{l}}{l^{\cancel{2}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

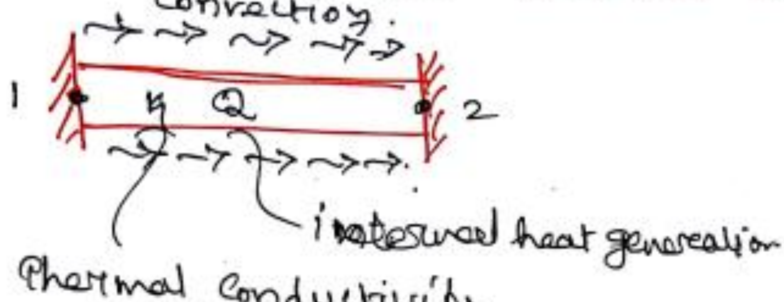
$$[K_c] = \frac{kA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

A = Area of the element, m^2

k = Thermal Conductivity of the element, W/m K

l = Length of the element, m.

(ii) One dimensional element with conduction, convection and internal heat generation.



Thermal conductivity.

Heat conduction part of stiffness matrix K_c for 1D element is $K_c = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Heat convection part of the stiffness matrix $[K]_h$ for 1D element is

$$[K_h] = \iint_S h[N]^T[N]\, dS \qquad dS = P \times dx$$

P = perimeter of the element.

$$= hP \int_0^l [N]^T[N]\, dx$$

$$= hP \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ x/l \end{Bmatrix} \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} dx$$

$$= hP \int_0^l \begin{bmatrix} \left(\frac{l-x}{l}\right)^2 & \frac{l-x}{l} \times \frac{x}{l} \\ \frac{x}{l} \times \left(\frac{l-x}{l}\right) & \frac{x}{l} \times \frac{x}{l} \end{bmatrix} dx$$

$$= hp \int_0^l \begin{bmatrix} \left(1-\frac{x}{l}\right)^2 & \frac{x}{l} - \frac{x^2}{l^2} \\ \frac{x}{l} - \frac{x^2}{l^2} & \frac{x^2}{l^2} \end{bmatrix} dx$$

$$= hp \begin{bmatrix} \frac{\left(1-\frac{x}{l}\right)^3}{3\times(-1/l)} & \frac{x^2}{2l} - \frac{x^3}{3l^2} \\ \frac{x^2}{2l} - \frac{x^3}{3l^2} & \frac{x^3}{3l^2} \end{bmatrix}_0^l$$

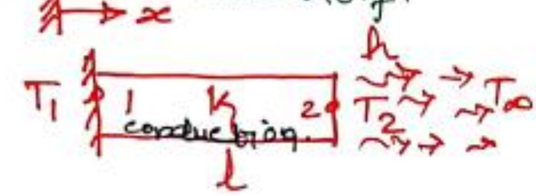
$$= hp \begin{bmatrix} \frac{\left(1-\frac{l}{l}\right)^3}{-3/l} - \frac{1^3}{-3/l} & \frac{l^2}{2l} - \frac{l^3}{3l^2} - 0 \\ \frac{l^2}{2l} - \frac{l^3}{3l^2} - 0 & \frac{l^3}{3l^2} - 0 \end{bmatrix}$$

$$= hp \begin{bmatrix} l/3 & l/2 - l/3 \\ l/2 - l/3 & l/3 \end{bmatrix}$$

$$= hp \begin{bmatrix} l/3 & l/6 \\ l/6 & l/3 \end{bmatrix}$$

$$[K_h] = \frac{hpl}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(i) One dimensional heat conduction with free end convection.



* Assume convection occurs only from the right end of the element.

One dimensional heat conduction element is given by

$$[K_c] = \frac{KA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

The convection term condition to the stiffness matrix is given by.

$$[K_h]_{end} = \iint_A h\,[N]^T[N]\,dA$$

↳ Shape factor.
↳ Heat transfer coefficient, W/m²K

$$[N] = [N_1 \;\; N_2] = \begin{bmatrix}\frac{l-x}{l} & \frac{x}{l}\end{bmatrix}$$

At node ② $x = l$

$$[N] = [N_1 \;\; N_2] = [0 \;\; 1] \quad [N]^T = \begin{bmatrix}0\\1\end{bmatrix}$$

Substitute $[N]$ and $[N]^T$ values in equation.

$$[K_h]_{end} = \iint_A h\begin{bmatrix}0\\1\end{bmatrix}[0 \;\; 1]\,dA$$

$$= h\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\int dx$$

$$= hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & hA\end{bmatrix}$$