Convolution Theorem

If $F(z)$ and $G(z)$ are the Z-transforms of $f(n)$ and $g(n)$ respectively, then

$Z\{f(n) * g(n)\} = F(z)\,G(z)$, where $f(n) * g(n)$

is defined as the convolution of $f(n)$ and $g(n)$ given by $f(n) * g(n) = \sum_{k=0}^{n} f(k)\, g(n-k)$.

Proof

We have $F(z)\, G(z) = \left[\sum_{n=0}^{\infty} f(n) z^{-n}\right] \left[\sum_{n=0}^{\infty} g(n) z^{-n}\right]$

$$F(z)\, G(z) = \left[f(0) + f(1) z^{-1} + \cdots + f(n) z^{-n} + \cdots\right]$$

$$\left[g(0) + g(1) z^{-1} + \cdots + g(n) z^{-n} + \cdots\right]$$

$$= \sum_{n=0}^{\infty} \left[f(0) g(n) + f(1) g(n-1) + \cdots + f(n) g(0)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} A_n z^{-n}$$

where $A_n = f(0) g(n) + \cdots + f(n) g(0)$

$$= \sum_{k=0}^{n} f(k)\, g(n-k)$$

$$= f(n) * g(n)$$

$$F(z)\, G(z) = \sum_{n=0}^{\infty} A_n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(f(n) * g(n)\right) z^{-n}$$

$$F(z)\, G(z) = Z\left\{f(n) * g(n)\right\}$$

Inverse Z - transform

If $Z[f(n)] = F(z)$, then $f(n)$ is called the inverse Z - transform of $F(z)$ and is denoted by $f(n) = Z^{-1}\{F(z)\}$.

Inverse Z - transform using Convolution Theorem

① Using convolution theorem, find

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}.$$

WKT $Z[a^n] = \dfrac{z}{z-a} \Rightarrow Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$

Let $F(z) = \dfrac{z}{z-a}$ and $G(z) = \dfrac{z}{z-b}$

$$\therefore Z^{-1}[F(z)G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = a^n * b^n$$

$$= \sum_{k=0}^{n} a^k \, b^{n-k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right].$$

$$= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\left(\frac{a}{b}\right) - 1} \right]$$

$$= b^n \left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}} \right]$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

② Using convolution theorem, evaluate

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right].$$

WKT $Z^{-1}\left[\frac{z}{z-1}\right] = 1^n$ and $Z^{-1}\left[\frac{z}{z-3}\right] = 3^n$.

$$\therefore Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1^n * 3^n$$

$$= \sum_{k=0}^{n} 1^k \cdot 3^{n-k}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + \cdots + 3 + 1$$

$$= \frac{3^{n+1} - 1}{3 - 1} = \frac{3^{n+1} - 1}{2}$$

③ Find the inverse Z-transform of $\dfrac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$ using convolution theorem.

WKT $Z^{-1}\left[\dfrac{z}{z-\frac{1}{2}}\right] = \left(\frac{1}{2}\right)^n \quad Z^{-1}\left[\dfrac{z}{z-\frac{1}{4}}\right] = \left(\frac{1}{4}\right)^n$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n-k}\left(\frac{1}{4}\right)^{k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)^{2k}}{\left(\frac{1}{2}\right)^{k}}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k}$$

$$= \left(\frac{1}{2}\right)^n \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} \left[1 - \left(\frac{1}{2}\right)^{n+1}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n}$$

④ Find the inverse Z-transform of $\frac{8z^2}{(2z-1)(4z-1)}$ using convolution theorem.

$$\frac{8z^2}{(2z-1)(4z-1)} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

⑤ Find the inverse Z-transform of $\frac{z}{(z+1)^2}$ by power series method.

Let $F(z) = \frac{z}{(z+1)^2} = \frac{z}{z^2(1+\frac{1}{z})^2} = \frac{1}{z}\left[1+\frac{1}{z}\right]^{-2}$

$$F(z) = \frac{1}{z}\left[1 - \frac{2}{z} + \frac{3}{z^2} - \frac{4}{z^3} + \cdots\right]$$

$$= \frac{1}{z} - \frac{2}{z^2} + \frac{3}{z^3} - \frac{4}{z^4} + \cdots$$

$$= z^{-1} - 2z^{-2} + 3z^{-3} - \cdots$$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} n(-1)^{n+1} z^{-n}$$

$$Z\{f(n)\} = Z\left[n(-1)^{n+1}\right]$$

$$f(n) = n(-1)^{n+1}$$