Inverse Z-transform by using Partial fraction

① Find the inverse Z-transform of $\dfrac{z^2+z}{(z-1)(z^2+1)}$

Let $F(z) = \dfrac{z^2+z}{(z-1)(z^2+1)} = \dfrac{Az}{z-1} + \dfrac{Bz^2+Cz}{z^2+1}$

$z^2+z = Az(z^2+1) + (Bz^2+Cz)(z-1)$

Put $z = 1$

$2 = 2A$

$\boxed{A = 1}$

Put $z = 0$

$0 = A - C$

$1 - C = 0$

$\boxed{C = 1}$

Coeff. of z^3

$0 = A + B$

$1 + B = 0$

$\boxed{B = -1}$

coeff. of z

$1 = A - C$

$1 = 1 - C$

$\boxed{C = 0}$

$\therefore \quad F(z) = \dfrac{z}{z-1} - \dfrac{z^2}{z^2+1}$

$Z^{-1}[F(z)] = Z^{-1}\left[\dfrac{z}{z-1}\right] - Z^{-1}\left[\dfrac{z^2}{z^2+1}\right]$

$= 1^n - \cos\dfrac{n\pi}{2} = 1 - \cos\dfrac{n\pi}{2}.$

2) Find the inverse Z-transform of $\dfrac{z}{(z-1)^2(z+1)}$

Let $\dfrac{F(z)}{z} = \dfrac{1}{(z-1)^2(z+1)} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-1)^2} + \dfrac{C}{(z+1)}$

$$1 = A(z+1)(z-1) + B(z+1) + C(z-1)^2$$

Put $z = 1$

$1 = 2B$

$\boxed{B = \frac{1}{2}}$

Put $z = -1$

$1 = 4C$

$\boxed{C = \frac{1}{4}}$

Put $z = 0$

$1 = -A + B + C$

$A = \frac{1}{2} + \frac{1}{4} - 1$

$\boxed{A = -\frac{1}{4}}$

$$\frac{F(z)}{z} = \frac{-1/4}{z-1} + \frac{1/2}{(z-1)^2} + \frac{1/4}{z+1}$$

$$F(z) = -\frac{1}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2} + \frac{1}{4}\frac{z}{z+1}$$

$$Z^{-1}[F(z)] = -\frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{2}Z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right]$$

$$= -\frac{1}{4}1^n + \frac{1}{2}n + \frac{1}{4}(-1)^n$$

③ Find the inverse Z-transform of $\frac{2z^2+3z}{(z+2)(z-4)}$.

Let $\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$

$$2z+3 = A(z-4) + B(z+2)$$

Put $z = 4$

$11 = 6B$

$$\boxed{B = \frac{11}{6}}$$

Put $z = -2$

$-1 = -6A$

$$\boxed{A = \frac{1}{6}}$$

$$\frac{F(z)}{z} = \frac{1/6}{z+2} + \frac{11/6}{z-4}$$

$$F(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

$$Z^{-1}[F(z)] = \frac{1}{6} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} Z^{-1}\left[\frac{z}{z-4}\right]$$

$$= \frac{1}{6}(-2)^n + \frac{11}{6} 4^n$$

Inverse Z-transform by Inverse integral method

① Find the inverse Z-transform of $\frac{z^2-3z}{(z+2)(z-5)}$

Let $F(z) = \frac{z^2-3z}{(z+2)(z-5)}$

The poles are gn. by $(z+2)(z-5) = 0$

$\Rightarrow z = -2, 5$ (simple poles)

Residue of $F(z) z^{n-1}$ at $z = -2$

$$R_1 = \lim_{z \to -2} (z+2) \frac{z^2-3z}{(z+2)(z-5)} z^{n-1} = -\frac{10}{7}(-2)^{n-1}$$

$$R_1 = \frac{5}{7}(-2)^n$$

Residue of $F(z)\, z^{n-1}$ at $z = 5$

$$R_2 = \lim_{z \to 5} (z-5) \frac{(z^2-3z)}{(z+2)(z-5)} z^{n-1}$$

$$= \frac{10}{7}(5)^{n-1} = \frac{2}{7}(5)^n$$

$$\therefore Z^{-1}\{F(z)\} = \frac{5}{7}(-2)^n + \frac{2}{7}(5)^n$$

(2) Find the inverse Z-transform of

$$\frac{(z^2+z)}{(z-2)^3}$$

Let $F(z) = \dfrac{z^2+z}{(z-2)^3}$

$$F(z)\, z^{n-1} = \frac{z^n(z+1)}{(z-2)^3}$$

The poles are gn. by $(z-2)^3 = 0$

$\Rightarrow z = 2$ (order 3)

Residue of $F(z)\, z^{n-1}$ at $z = 2$

$$R_1 = \frac{1}{2!} \lim_{z \to 2} \frac{d^2}{dz^2} \frac{z^n(z+1)}{(z-2)^3} (z-2)^3$$

$$= \frac{1}{2!} \lim_{z \to 2} \frac{d^2}{dz^2} z^n(z+1)$$

$$= \frac{1}{2} \lim_{z \to 2} \frac{d}{dz} \left[z^n + (z+1)\, n z^{n-1} \right]$$

$$= \frac{1}{2} \lim_{z \to 2} \left[n z^{n-1} + n \left\{ (z+1)(n-1) z^{n-2} + z^{n-1} \right\} \right]$$

$$= \frac{1}{2} \lim_{z \to 2} \left[n z^{n-1} + (z+1) n(n-1) z^{n-2} + n z^{n-1} \right]$$

$$= \frac{1}{2} \lim_{z \to 2} \left[n z^{n-1} + n(n-1) z^{n-1} - n \right.$$

$$= \frac{1}{2} \lim_{z \to 2} \left[n z^{n-1} + n(n-1) z^{n-2} + n(n-1) z^{n-1} + n z^{n-1} \right]$$

$$= \frac{1}{2} \lim_{z \to 2} \left[z^{n-1} (n + n^2 - \cancel{n} + \cancel{n}) + n(n-1) z^{n-2} \right]$$

$$= \frac{1}{2} \left[(n^2+n) 2^{n-1} + (n^2-n) 2^{n-2} \right]$$

$$= \frac{1}{2} 2^{n-2} \left[2n^2 + 2n + n^2 - n \right]$$

$$= (3n^2+n) 2^{n-3} = \left[\frac{3}{8} n^2 + \frac{n}{8} \right] 2^n$$

$$\therefore Z^{-1} \{ F(z) \} = \left[\frac{3}{8} n^2 + \frac{n}{8} \right] 2^n$$

inverse Z ~~transform~~

③ Find the residue of $\dfrac{z^2+z}{(z-1)^2}$.

Let $F(z) = \dfrac{z^2+z}{(z-1)^2}$

$$F(z) z^{n-1} = \frac{z^n (z+1)}{(z-1)^2}$$

The poles are gn. by $(z-1)^2 = 0$

$z = 1$ (order 2)

The residue of $F(z)\, z^{n-1}$ at $z = 1$

$$= \lim_{z \to 1} \frac{d}{dz} \frac{z^n (z+1)}{(z-1)^2} (z-1)^2$$

$$= \lim_{z \to 1} \left[n z^{n-1}(z+1) + z^n \right]$$

$$= \left[n(2) + 1 \right] = \underline{\underline{2n+1}}$$

$$\therefore Z^{-1}\{F(z)\} = \underline{\underline{2n+1}}$$