Final Value Theorem

If $Z\{f(n)\} = F(z)$, then

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1)\,F(z).$$

Proof :

By defn, $Z\{f(n+1) - f(n)\} = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$

$$Z\{f(n+1)\} - Z\{f(n)\} = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$$

$$z\left[F(z) - f(0)\right] - F(z) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$$

$$F(z)(z-1) - z f(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$$

Taking limit as $z \to 1$,

$$\lim_{z\to 1}\left[(z-1)F(z)\right] - f(0) = \lim_{z\to 1}\sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} [f(n+1) - f(n)]$$

$$= \lim_{n\to\infty}\left[f(1) - f(0) + f(2) - f(1) + \dots + f(n+1) - f(n)\right]$$

$$= \lim_{n\to\infty}\left[f(n+1)\right] - f(0)$$

$$\lim_{z \to 1} \left[(z-1)\, F(z) \right] = \lim_{n \to \infty} f(n+1)$$

$$\lim_{z \to 1} \left[(z-1)\, F(z) \right] = \lim_{n \to \infty} f(n)$$

Unit Sample sequence $\delta(n)$

The unit sample sequence $\delta(n)$ is defined as the sequence with values.

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

Unit step Sequence

The unit step sequence

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

1) Find the Z - transform of

(i) $x(n) = a^n$ (ii) $x(n) = n$ (iii) $x(n) = n^2$

(iv) $x(n) = \cos n\alpha$ (v) $x(n) = \sin n\alpha$

(vi) $x(n) = r^n \cos n\alpha$ (vii) $x(n) = r^n \sin n\alpha$.

(i) $Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$

$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots$

$= \left[1 - \frac{a}{z}\right]^{-1} = \frac{1}{1 - a/z} = \frac{z}{z-a}$

(ii) $Z[n] = \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$

$= \cancel{\Sigma} \left[\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots\right]$

$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right]$

$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$

$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2} = \frac{1}{z}\left[\frac{z}{z-1}\right]^2$

$$\boxed{Z[n] = \frac{z}{(z-1)^2}}$$

(iii) $Z[n^2] = \sum_{n=0}^{\infty} n^2/z^{-n}$.

$= \cancel{\sum_{n=0}^{\infty}}$

$Z[n^2] = Z[n \cdot n] = (-z)\frac{d}{dz} Z[n]$

$= (-z)\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$

$$= (-z)\left[\frac{(z-1)^2 - z\,2(z-1)}{(z-1)^4}\right]$$

$$= (-z)\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

(iv) WKT $Z[a^n] = \frac{z}{z-a}$

$$Z\left[e^{in\omega}\right] = \frac{z}{z-e^{i\omega}} = \frac{z}{z-\cos\omega - i\sin\omega}$$

$$Z\left[\cos n\omega + i\sin n\omega\right] = \frac{z\left[z-\cos\omega + i\sin\omega\right]}{(z-\cos\omega)^2 + \sin^2\omega}$$

$$= \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1} + \frac{i\,z\sin\omega}{z^2-2z\cos\omega+1}$$

$$\Rightarrow Z\left[\cos n\omega\right] = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}$$

(v) $$Z\left[\sin n\omega\right] = \frac{z\sin\omega}{z^2-2z\cos\omega+1}$$

(vi)

$$Z\left[r^n \cos n\omega\right] = \frac{\frac{z}{r}\left(\frac{z}{r} - \cos\omega\right)}{\left(\frac{z}{r}\right)^2 - 2\left(\frac{z}{r}\right)\cos\omega + 1}$$

$$= \frac{z\,(z - r\cos\omega)}{z^2 - 2zr\cos\omega + r^2}$$

(vii)

$$Z\left[r^n \sin n\omega\right] = \frac{\frac{z}{r}\sin\omega}{\left(\frac{z}{r}\right)^2 - 2\left(\frac{z}{r}\right)\cos\omega + 1}$$

$$= \frac{zr\sin\omega}{z^2 - 2zr\cos\omega + r^2}$$

② Find the Z-transform of (i) $\frac{1}{n}$, $n>0$

(ii) $\cos\left(\frac{n\pi}{2}\right)$ (iii) $\frac{1}{n(n+1)}$ where $n \neq -1$

(i)

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$$

$$= -\log\left[1 - \frac{1}{z}\right] \quad \text{if } \left|\frac{1}{z}\right| < 1$$

$$= \log\left[\frac{z}{z-1}\right] \quad \text{if } |z| > 1$$

(ii)

$$Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{2}\right) z^{-n}$$

$$= 1 - \frac{1}{z^2} + \frac{1}{z^4} - \cdots$$

$$= \left[1+\frac{1}{z^2}\right]^{-1} = \frac{z^2}{z^2+1} \quad \text{if } |z|>1$$

(iii) $$Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$= Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right]$$

$$= \log\left(\frac{z}{z-1}\right) - \left[1+\frac{1}{2z}+\frac{1}{3z^2}+\cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) - z\left[\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) + z\log\left(1-\frac{1}{z}\right)$$

$$= \log\left(\frac{z}{z-1}\right) + z\log\left(\frac{z-1}{z}\right)$$

$$= (z-1)\log\left(\frac{z-1}{z}\right)$$

(3) Find the Z-transform of $\frac{1}{(n+1)(n+2)}$

$$Z\left[\frac{1}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$$

$$= Z\left[\frac{1}{n+1}\right] - Z\left[\frac{1}{n+2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} \mp \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= \left[1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots\right] \mp \left[\frac{1}{2} + \frac{1}{3z} + \frac{1}{4z^2} + \cdots\right]$$

$$= z\left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots\right] \mp z^2\left[\frac{1}{2z^2} + \frac{1}{3z^3} + \cdots\right]$$

$$= z\left\{-\log\left(1 - \frac{1}{z}\right)\right\} \mp z^2\left\{-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z}\right\}$$

$$= z \log\left(\frac{z}{z-1}\right) + z^2\left[\log\left(\frac{z-1}{z}\right) + \frac{1}{z}\right]$$

$$= -z \log\left(\frac{z-1}{z}\right) + z^2 \log\left(\frac{z-1}{z}\right) + z$$

$$= z + (z^2 - z)\log\left(\frac{z-1}{z}\right)$$

(4) Find the Z - transform of (i) e^{-at} and (ii) $\cos \omega t$.

(i) $$Z\left[e^{-at}\right] = \sum_{n=0}^{\infty} e^{-anT} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-aT}\right)^n z^{-n}$$

$$= \frac{z}{z - e^{-aT}}$$

(ii) $Z[\cos\omega t] = Z\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$

$$= \frac{1}{2} Z\left[e^{i\omega t} + e^{-i\omega t}\right]$$

$$= \frac{1}{2}\left\{ Z[e^{i\omega t}] + Z[e^{-i\omega t}] \right\}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\omega T}} + \frac{z}{z-e^{-i\omega T}}\right]$$

$$= \frac{z}{2}\left\{\frac{1}{z-e^{i\omega T}} + \frac{1}{z-e^{-i\omega T}}\right\}$$

$$= \frac{z}{2}\left\{\frac{z-e^{-i\omega T} + z - e^{i\omega T}}{(z-e^{i\omega T})(z-e^{-i\omega T})}\right\}$$

$$= \frac{z}{2}\left\{\frac{2z-(e^{i\omega T}+e^{-i\omega T})}{z^2 - z(e^{i\omega T}+e^{-i\omega T})+1}\right\}$$

$$= \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$$

(5) Find the Z-transform of (i) $e^{3t}\cos t$ (ii) $t^2 e^{-2t}$.

(i) $Z[e^{3t}\cos t] = \left\{Z[\cos t]\right\}_{z \to ze^{-3T}}$

$$= \left[\frac{z(z-\cos T)}{z^2-2z\cos T+1}\right]_{z \to ze^{-3T}}$$

$$= \frac{ze^{-3T}\left[ze^{-3T}-\cos T\right]}{z^2e^{-6T}-2ze^{-3T}\cos T+1}$$

(ii) $Z\left[t^2e^{-2t}\right] = \left\{Z\left[t^2\right]\right\}_{z \to ze^{2T}}$

$$= \left[\frac{T^2z(z+1)}{(z-1)^3}\right]_{z \to ze^{2T}}$$

$$= \frac{T^2ze^{2T}\left[ze^{2T}+1\right]}{(ze^{2T}-1)^3}$$

$$= \frac{T^2ze^{4T}\left[z+e^{-2T}\right]}{e^{6T}(z-e^{-2T})^3}$$

$$= \frac{T^2ze^{-2T}\left[z+e^{-2T}\right]}{(z-e^{-2T})^3}$$