Total Derivative

1) If $u = f(x,y)$ then total differential of u is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

2) If $u = f(x,y)$ where $x = g_1(t)$, $y = g_2(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

3) If $u = f(x,y)$ where y is a function of x then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \text{and}$$

$$\frac{du}{dy} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dy} + \frac{\partial u}{\partial y}$$

Differentiation of Implicit function :

If $f(x,y) = c$ where c may be zero or non-zero is an implicit function of x & y then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}.$$

Type. I

1) If $u = x^2 + y^2$ then find total differential of u.

Soln: $u = x^2 + y^2$

$\frac{\partial u}{\partial x} = 2x$,

$\frac{\partial u}{\partial y} = 2y$.

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} \cdot dy$

$= 2x\, dx + 2y\, dy$.

Type. II

(3) Find $\frac{du}{dt}$ if $u = x^2 + y^2 + z^2$ where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$.

Soln:

$u = x^2 + y^2 + z^2$.

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial u}{\partial z} = 2z$.

Gn: $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

$\frac{dx}{dt} = e^t$ $\frac{dy}{dt} = e^t \cos t + e^t \sin t$ $\frac{dz}{dt} = -e^t \sin t + e^t \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

$$= 2x\cdot e^t + 2y\cdot(e^t\cos t + e^t\sin t) + 2z\cdot(e^t\cos t - e^t\sin t)$$

$$= 2e^t e^t + 2e^t\sin t\,(e^t\cos t + e^t\sin t) + 2e^t\cos t\,(e^t\cos t - e^t\sin t)$$

$$= 2e^{2t} + \cancel{2e^{2t}\sin t\cos t} + 2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t - \cancel{2e^{2t}\cos t\sin t}$$

$$= 2e^{2t} + 2e^{2t} = 4e^{2t}.$$

Type -

(iv) If $x^3+y^3 = 3axy$ find $\frac{dy}{dx}$

Soln: $x^3+y^3-3axy=0$

Let $f(x,y) = x^3+y^3-3axy$

$$\frac{\partial f}{\partial x} = 3x^2-3ay$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax$$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{3(x^2-ay)}{3(y^2-ax)} = \frac{ay-x^2}{y^2-ax}.$$