MAXIMA OR MINIMA

CRITICAL POINT OR STATIONARY POINT.

Critical point or stationary point of the function is a point (a,b) if $\frac{\partial f}{\partial x} = 0$ at (a,b).

NECESSARY CONDITION

The necessary condition for the function $f(x,y)$ to have a ~~max. or~~ maxima or minima at the point (a,b) is $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ at (a,b)

SUFFICIENT CONDITION

Let $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x \partial y}$, $C = \frac{\partial^2 f}{\partial y^2}$

i) If $AC - B^2 > 0$, $A > 0$	Minimum value.
ii) If $AC - B^2 > 0$, $A < 0$	Maximum value.
iii) If $AC - B^2 < 0$,	Neither maximum nor minimum (saddle point)
iv) If $AC - B^2 = 0$	Inconclusive.

1. Find the maxima & minima of function

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$

$\frac{\partial f}{\partial y} = 6xy - 30y$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$ — ①

$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 30y = 0$ — ②

② $\Rightarrow 6y(x-5) = 0$

$y = 0 \quad x - 5 = 0$

$x = 5$

Put $y = 0$ in ①

$3x^2 - 30x + 72 = 0$

$x^2 - 10x + 24 = 0$

$x = 4, 6$

$24 \to -6, -4$

$\therefore$ The point is $(4,0), (6,0)$.

Put $x = 5$ in ①

$75 + 3y^2 - 150 + 70 = 0$

$3y^2 - 3 = 0$

$y^2 = 1$

$y = \pm 1$

The point is $(5,-1), (5,1)$

The critical points are

$(4,0), (6,0), (5,-1), (5,1)$

$$A = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6y.$$

$$C = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

Critical point	A	B	C	$AC-B^2$	Conclusion.
x y (6,0)	6	0	6	36	Minimum value
(4,0)	-6	0	-6	36	Maximum value
(5,1)	0	6	0	-36	Saddle point.
(5,-1)	0	-6	0	-36	Saddle point

Maximum value $= f(4,0) = 112$

Minimum value $= f(6,0) = 108.$

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2. $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x$$

$$\frac{\partial f}{\partial y} = 6xy - 6y$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 6x = 0$$

$$x^2 + y^2 - 2x = 0 \quad -(1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 6y = 0$$

$$xy - y = 0 \quad -(2)$$

$$(2) \Rightarrow y(x-1) = 0$$

$$y = 0 \qquad x - 1 = 0$$

$$x = 1$$

Put $y = 0$ in (1)

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0 \qquad x = 2$$

$\therefore$ The point is $(0,0)$, $\cancel{(0,2)}$ $(2,0)$

Put $\cancel{y=}$ $x = 1$ in (1)

$$1 + y^2 - 2 = 0$$

$$y^2 - 1 = 0$$

$$y^2 = 1 \quad \Rightarrow y = \pm 1$$

$\therefore$ The point is $(1,1)$, $(1,-1)$

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The critical point is $(0,0)$, $(2,0)$, $(1,1)$, $(1,-1)$.

$A = f_{xx} = 6x - 6$

$B = f_{xy} = 6y$

$C = f_{yy} = 6x - 6$

Critical Point	A	B	C	$AC-B^2$	Conclusion
$(0,0)$	-6	0	-6	36	Maximum Value
$(2,0)$	6	0	6	36	Minimum Value
$(1,1)$	0	6	0	-36	saddle point.
$(1,-1)$	0	-6	0	-36	Saddle point.

Maximum value $= f_{(0,0)} = 4$.

Minimum value $= f_{(2,0)} = 8 - 12 + 4$
$= 0$

3. $f = x^3 + y^3 - 12x - 3y + 20.$

$$\frac{\partial f}{\partial x} = 3x^2 - 12$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3$$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 12 = 0$ — (1)

$3(x^2-4) = 0 \Rightarrow x^2 - 4 = 0$ — $\cancel{(1)}$

$x^2 = 4 \Rightarrow x = \pm 2$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 3 = 0$

$y^2 - 1 = 0$ — (2)

(2) $\Rightarrow y^2 = 1$

$y = \pm 1$

~~put $y = 1$ in (1).~~

The critical points are $(2,1), (2,-1), (-2,1), (-2,-1)$

$A = f_{xx} = 6x$

$B = f_{xy} = 0.$

$C = f_{yy} = 6y$

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Critical point.	A	B	C	$AC-B^2$	Conclusion.
$(2,1)$	12	0	6	72	Minimum Value
$(2,-1)$	12	0	-6	-72	Saddle point.
$(-2,1)$	-12	0	6	-72	Saddle point.
$(-2,-1)$	-12	0	-6	72	Maximum value

$$\text{Maximum value} = f_{(-2,-1)} = -8-1+24+3+20$$

$$= 24 - 6 + 20$$

$$= \cancel{70}\ 20 + 18$$

$$= \cancel{110}\ 38$$

$$\text{Minimum value} = f_{(2,1)} = 8+1-24-3+20$$

$$= 9-4-3$$

$$= 2.$$