EULER'S THEOREM (2m)

HOMOGENEOUS FUNCTION

A function $f(x, y)$ is said to be homogeneous function of degree n if

$$\boxed{f(tx, ty) = t^n f(x, y)}$$

EULER'S STATEMENT

If $f(x,y)$ is a homogeneous function of degree n, then Euler's theorem states that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

condition

1. Prove: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ by using Euler's theorem where $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}$$

$$f = \tan u = \frac{x^3+y^3}{x-y}$$

$$f(tx, ty) = \frac{t^3x^3+t^3y^3}{tx-ty}$$

$$= \frac{t^3(x^3+y^3)}{t(x-y)}$$

$$f(tx,ty) = \frac{t^2(x^3+y^3)}{x-y}$$

It is a homogeneous function of degree 2

By Euler's theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f$$

$$x \frac{\partial (\tan u)}{\partial x} + y \frac{\partial (\tan u)}{\partial y} = 2 \tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cos u}$$

$$\sec^2 u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 2 \frac{\sin u}{\cos u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cos u} \times \frac{1}{\sec^2 u}$$

$$= 2 \frac{\sin u}{\cos u} \times \cos^2 u$$

$$= 2 \sin u \cos u$$

$$= \sin 2u$$

Hence proved.

2. P.T $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$ where

$$u = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$$

$$\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

let $f(x,y) = \sin u = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$$f(tx,ty) = \frac{\sqrt{tx}-\sqrt{ty}}{\sqrt{tx}+\sqrt{ty}}$$

$$= \frac{\sqrt{t}(\sqrt{x}-\sqrt{y})}{\sqrt{t}(\sqrt{x}+\sqrt{y})}$$

$$f(tx,ty) = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = t^0 f(x,y)$$

∴ It is a homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial}{\partial y}(\sin u) = 0(\sin u)$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0$$

$$\cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Hence proved.

3. Verify Euler's Theorem for $U = x^3 \sin\left(\frac{y}{x}\right)$

$$U(tx, ty) = t^3 x^3 \sin\left(\frac{ty}{tx}\right)$$

$$= t^3 x^3 \sin\left(\frac{y}{x}\right)$$

$$= t^3 U$$

$\therefore$ U is a homogeneous function of degree 3

~~3. Verify Euler's Theorem for $U = x^3 \sin\left(\frac{y}{x}\right)$~~

By Euler's Theorem,

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = nU$$

L.H.S.

$$\frac{\partial U}{\partial x} = 3x^2 \sin\left(\frac{y}{x}\right) + x^3 \cos\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$x \frac{\partial U}{\partial x} = 3x^3 \sin\left(\frac{y}{x}\right) + x^{\cancel{4}\,2} \cos\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= 3x^3 \sin\left(\frac{y}{x}\right) + \cancel{3x}\, x^2 \cos\left(\frac{y}{x}\right)(-y)$$

$$x \frac{\partial U}{\partial x} = 3x^3 \sin\left(\frac{y}{x}\right) - x^2 y \cos\left(\frac{y}{x}\right)$$

$$\frac{\partial U}{\partial y} = x^{\cancel{3}\,2} \cancel{\sin} \cos\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^2 \cos\left(\frac{y}{x}\right)$$

$$y \frac{\partial U}{\partial y} = x^2 y \cos\left(\frac{y}{x}\right)$$

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$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3 \sin\frac{y}{x} - x^2 y\cos\left(\frac{y}{x}\right) + x^2 y\cos\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3 \sin\frac{y}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$$

Hence Verified.