MAXIMA & MINIMA OF FUNCTIONS OF TWO VARIABLES

Conditions for $f(x,y)$ to be maximum (or) minimum

(i) Necessary condition :

The necessary condition for $f(x,y)$ to have a maximum or minimum at the point (a,b) are $\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = 0$ & $\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = 0$

(ii) Sufficient condition :

If $f_x(a,b) = 0$, $f_y(a,b) = 0$, $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$ then.

(1) $f(a,b)$ is maximum value if $AC - B^2 > 0$ & $A < 0$ and the point (a,b) is called the maximum point.

(2) $f(a,b)$ is minimum value if $AC - B^2 > 0$ & $A > 0$ and the point (a,b) is called the minimum point.

(3) $f(a,b)$ is neither maximum nor minimum

(a) not an extremum if $AC - B^2 < 0$ and the point (a,b) is called saddle point.

(4) If $AC - B^2 = 0$ then the test is inconclusive.

Stationary points:-

A function $f(x,y)$ is said to be stationary at the point (a,b) if $f_x = 0$, $f_y = 0$

Notations:

$$\frac{\partial f}{\partial x} = f_x ; \quad \frac{\partial f}{\partial y} = f_y ; \quad \frac{\partial^2 f}{\partial x^2} = f_{xx} ; \quad \frac{\partial^2 f}{\partial y^2} = f_{yy} ; \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

1) Find the maximum and minimum value of

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Soln:

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$$f_x = 3x^2 + 3y^2 - 6x$$

$$f_y = 6xy - 6y$$

$$A: \; f_{xx} = 6x - 6$$

$$B: \; f_{xy} = 6y$$

$$C: \; f_{yy} = 6x - 6$$

To find Stationary points:

$f_x = 0$

$\Rightarrow 3x^2+3y^2-6x = 0$

$\Rightarrow x^2+y^2-2x = 0$

$\Rightarrow y^2 = 2x-x^2$

$f_y = 0$

$6xy-6y = 0$

$6y(x-1) = 0$

$\Rightarrow y = 0$ or $x-1 = 0$

$\Rightarrow y = 0$ or $x = 1$.

when $x = 1$;

$y^2 = 1$

$\Rightarrow y = \pm 1$

when $y = 0$

$\Rightarrow 0 = 2x-x^2$

$\Rightarrow x(2-x) = 0$

$\Rightarrow x = 0,\ x = 2$.

$\therefore$ The stationary points are. $(0,0), (2,0), (1,1), (1,-1)$

Stationary points	A $f_{xx}=6x-6$	B $f_{xy}=6y$	C $f_{yy}=6x-6$	$AC-B^2$	Conclusion
$(0,0)$	$-6<0$	0	-6	$36>0$	Maximum point
$(2,0)$	$6>0$	0	6	$36>0$	Minimum point
$(1,1)$	0	6	0	$-36<0$	Saddle point
$(1,-1)$	0	-6	0	$-36<0$	Saddle point

To find maximum value:

Maximum value of $f(x,y)$ at the point $(0,0)$ is

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$f(0,0) = 4$, a maximum value.

To find minimum value:

Minimum value of $f(x,y)$ at the point $(2,0)$ is

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$$f(2,0) = 8 + 0 - 12 - 0 + 4$$

$$= 0, \text{ a minimum value.}$$

(2) Find the max. & min. value of $f(x,y) = x^2 - xy + y^2 - 2x + y$

$$f(x,y) = x^2 - xy + y^2 - 2x + y.$$

$$f_x = 2x - y - 2.$$

$$f_y = -x + 2y + 1$$

A: $f_{xx} = 2.$

B: $f_{xy} = -1$

C: $f_{yy} = 2$

To find stationary points:

$f_x = 0 \Rightarrow 2x - y - 2 = 0$

$\Rightarrow 2x - 2 = y \quad \text{--- (1)}$

$f_y = 0 \Rightarrow -x + 2y + 1 = 0$

$\Rightarrow 2y = x - 1$

$\Rightarrow y = \frac{x-1}{2} \quad \text{--- (2)}$

From (1) & (2) $\Rightarrow 2x - 2 = \frac{x-1}{2}$

$\Rightarrow 4x - 4 = x - 1$

$\Rightarrow 3x = 3$

$\boxed{x = 1}$

when $x = 1$ in $2x - 2 = y$

$\Rightarrow \boxed{y = 0}$

$\therefore$ The stationary point is $(1, 0)$.

Stationary point	A $f_{xx} = 2$	B $f_{xy} = 1$	C $f_{yy} = 2$	$AC - B^2$	Conclusion
$(1, 0)$	$2 > 0$	1	2	$3 > 0$	minimum point.

To find minimum value:

$f(x, y) = x^2 - xy + y^2 - 2x + y$

$f(1, 0) = 1 - 2 = -1$, a minimum value.

③ $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Soln: $f_x = 3x^2 - 3$

$f_y = 3y^2 - 12$

A: $f_{xx} = 6x$

C: $f_{yy} = 6y$

B: $f_{xy} = 0$

To find stationary points

$f_x = 0$; $f_y = 0$

$\Rightarrow 3x^2 - 3 = 0$; $3y^2 - 12 = 0$

$\Rightarrow x = \pm 1$; $y = \pm 2$.

$\therefore$ The stationary pts are $(1,2), (1,-2), (-1,2), (-1,-2)$

Stationary point	A $f_{xx} = 6x$	B $f_{xy} = 0$	C $f_{yy} = 6y$	$AC - B^2$	Conclusion
$(1,2)$	$6 > 0$	0	12	$72 > 0$	minimum pt.
$(1,-2)$	$6 > 0$	0	-12	$-72 < 0$	Saddle point
$(-1,2)$	$-6 < 0$	0	12	$-72 < 0$	Saddle point
$(-1,-2)$	$-6 < 0$	0	-12	$72 > 0$	Maximum point

To find maximum value :

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f(-1,-2) = 38$, a max. value.

To find mini. value :

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f(1,2) = 2$, a mini. value.