



DEPARTMENT OF MATHEMATICS

UNIT - IV FUNCTIONS OF SEVERAL VARIABLES

MAXIMA & MINIMA OF FUNCTIONS OF TWO VARIABLES

Conditions for $f(x, y)$ to be maximum (or) minimum

(i) Necessary condition :

The necessary condition for $f(x, y)$ to have a maximum or minimum at the point (a, b) are $\frac{\partial f}{\partial x}(a, b) = f_x(a, b) = 0$ & $\frac{\partial f}{\partial y}(a, b) = f_y(a, b) = 0$

(ii) Sufficient condition :

If $f_x(a, b) = 0$, $f_y(a, b) = 0$, $f_{xx}(a, b) = A$,
 $f_{xy}(a, b) = B$, $f_{yy}(a, b) = C$ then,

(1) $f(a, b)$ is maximum value if $AC - B^2 > 0$ & $A < 0$ and the point (a, b) is called the maximum point.

(2) $f(a, b)$ is minimum value if $AC - B^2 > 0$ & $A > 0$ and the point (a, b) is called the minimum point.

(3) $f(a, b)$ is neither maximum nor minimum
(a) not an extremum if $AC - B^2 < 0$ and the point (a, b) is called saddle point.

(4) If $AC - B^2 = 0$ then the test is inconclusive.



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Stationary points:-

A function $f(x, y)$ is said to be stationary at the point (a, b) if $f_x = 0$, $f_y = 0$.

Notations:

$$\frac{\partial f}{\partial x} = f_x; \quad \frac{\partial f}{\partial y} = f_y; \quad \frac{\partial^2 f}{\partial x^2} = f_{xx}; \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}; \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

1) Find the maximum and minimum value of

$$f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Soln: $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$$f_x = 3x^2 + 3y^2 - 6x$$

$$f_y = 6xy - 6y$$

$$A: f_{xx} = 6x - 6$$

$$B: f_{xy} = 6y - 6$$

$$C: f_{yy} = 6x - 6$$



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To find Stationary points:

$$f_x = 0$$

$$\Rightarrow 3x^2 + 3y^2 - 6x = 0$$

$$\Rightarrow x^2 + y^2 - 2x = 0$$

$$\Rightarrow y^2 = 2x - x^2$$

when $x = 1$;

$$y^2 = 1$$

$$\Rightarrow y = \pm 1$$

when $y = 0$;

$$\Rightarrow 0 = 2x - x^2$$

$$\Rightarrow x(2-x) = 0$$

$$\Rightarrow x = 0, x = 2$$

\therefore The stationary points are $(0, 0)$, $(2, 0)$, $(1, 1)$, $(1, -1)$

Stationary points	A	B	C	$Ac - B^2$	Conclusion
$(0, 0)$	$-6 < 0$	0	-6	$36 > 0$	Maximum point
$(2, 0)$	$6 > 0$	0	6	$36 > 0$	Minimum point
$(1, 1)$	0	6	0	$-36 < 0$	Saddle point
$(1, -1)$	0	-6	0	$-36 < 0$	Saddle point



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To find maximum value:

Maximum value of $f(x,y)$ at the point $(0,0)$ is

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$$f(0,0) = 4, \text{ a maximum value.}$$

To find minimum value:

Minimum value of $f(x,y)$ at the point $(2,0)$ is

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$$f(2,0) = 8 + 0 - 12 - 0 + 4$$

$$= 0, \text{ a minimum value.}$$

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② Find the max. & min. value of $f(x,y) = x^2 - xy + y^2 - 2x + y$

$$f(x,y) = x^2 - xy + y^2 - 2x + y.$$

$$f_x = 2x - y - 2.$$

$$f_y = -x + 2y + 1$$

$$A: f_{xx} = 2.$$

$$B: f_{xy} = -1$$

$$C: f_{yy} = 2$$



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To find stationary points:

$$\begin{aligned} f_x = 0 &\Rightarrow 2x - y - 2 = 0 \\ &\Rightarrow 2x - 2 = y \quad \text{--- (1)} \end{aligned}$$

$$\begin{aligned} f_y = 0 &\Rightarrow -x + 2y + 1 = 0 \\ &\Rightarrow 2y = x - 1 \\ &\Rightarrow y = \frac{x-1}{2} \quad \text{--- (2)} \end{aligned}$$

$$\begin{aligned} \text{From (1) \& (2)} &\Rightarrow 2x - 2 = \frac{x-1}{2} \\ &\Rightarrow 4x - 4 = x - 1 \\ &\Rightarrow 3x = 3 \\ &\quad \boxed{x = 1} \end{aligned}$$

$$\begin{aligned} \text{when } x = 1 \text{ in } 2x - 2 = y \\ &\Rightarrow \boxed{y = 0} \end{aligned}$$

∴ The stationary point is (1, 0).

Stationary point	A	B	C	AC - B ²	Conclusion
(1, 0)	$f_{xx} = 2$	$f_{yy} = 1$	$f_{xy} = 2$	$3 > 0$	Minimum point.

To find minimum value:

$$f(x, y) = x^2 - xy + y^2 - 2x + y$$

$$f(1, 0) = 1 - 2 = -1, \text{ a minimum value.}$$



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③ $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

Soln: $f_x = 3x^2 - 3$

$f_y = 3y^2 - 12$

A: $f_{xx} = 6x$

C: $f_{yy} = 6y$

B: $f_{xy} = 0$

To find stationary points

$f_x = 0$; $f_y = 0$

$\Rightarrow 3x^2 - 3 = 0$; $3y^2 - 12 = 0$

$\Rightarrow x = \pm 1$; $y = \pm 2$

\therefore The stationary pts are $(1, 2), (1, -2), (-1, 2), (-1, -2)$

Stationary point	A $f_{xx} = 6x$	B $f_{xy} = 0$	C $f_{yy} = 6y$	$AC - B^2$	Conclusion
$(1, 2)$	$6 > 0$	0	12	$72 > 0$	minimum pt.
$(1, -2)$	$6 > 0$	0	-12	$-72 < 0$	Saddle point
$(-1, 2)$	$-6 < 0$	0	12	$-72 < 0$	Saddle point
$(-1, -2)$	$-6 < 0$	0	-12	$72 > 0$	Maximum point



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To find maximum value :

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

$$f(-1, -2) = 38, \text{ a max. value.}$$

To find mini. value :

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

$$f(1, 2) = 2, \text{ a mini. value.}$$