METHOD OF LAGRANGIAN'S MULTIPLIERS

We can find an extreme value of the function $f(x,y,z)$ subject to the constrained $g(x,y,z)=0$

Define $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$ where λ is an undetermined constant called the Lagrangian multipliers.

By solving the eqn.

$$\frac{\partial F}{\partial x} = 0, \ \frac{\partial F}{\partial y} = 0, \ \frac{\partial F}{\partial z} = 0, \ \frac{\partial F}{\partial \lambda} = 0,$$ we get the values of x, y, z and λ. [using λ value find x, y, z] find the values either maximum or minimum by substituting x, y, z in $f(x,y,z)$.

1) Find the minimum value of $x^2+y^2+z^2$, given that $ax+by+cz=p$

Let $f = x^2+y^2+z^2$ and $g = ax+by+cz-p$

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$
$$= x^2+y^2+z^2 + \lambda[ax+by+cz-p]$$

$$\frac{\partial F}{\partial x} = 2x + a\lambda$$

$$\Rightarrow \frac{\partial F}{\partial x} = 0 \Rightarrow 2x + a\lambda = 0$$

$$\Rightarrow x = -\frac{\lambda a}{2}$$

$$\frac{\partial F}{\partial y} = 2y + b\lambda$$

$$\Rightarrow \frac{\partial F}{\partial y} = 0 \Rightarrow 2y + b\lambda = 0$$

$$\Rightarrow y = -\frac{\lambda b}{2}$$

$$\frac{\partial F}{\partial z} = 2z + c\lambda$$

$$\Rightarrow \frac{\partial F}{\partial z} = 0 \Rightarrow 2z + c\lambda = 0$$

$$\Rightarrow z = -\frac{\lambda c}{2}$$

$$\frac{\partial F}{\partial \lambda} = ax + by + cz - p.$$

$$\Rightarrow \frac{\partial F}{\partial \lambda} = 0 \Rightarrow ax + by + cz = p$$

$$\Rightarrow a \times -\lambda/_2 a + b \times -\lambda/_2 b + c \times -\lambda/_2 c = p$$

$$\Rightarrow -\frac{\lambda}{2}\left[a^2 + b^2 + c^2\right] = p$$

$$\Rightarrow \lambda = -\frac{2p}{a^2 + b^2 + c^2}.$$

$$\therefore x = -\frac{\lambda a}{2} \Rightarrow x = \frac{pa}{a^2+b^2+c^2}$$

$$y = -\frac{\lambda b}{2} \Rightarrow y = \frac{pb}{a^2+b^2+c^2}$$

$$z = -\frac{\lambda c}{2} \Rightarrow z = \frac{pc}{a^2+b^2+c^2}$$

sub x, y, z in f

$$f = x^2+y^2+z^2$$

$$= \left(\frac{pa}{a^2+b^2+c^2}\right)^2 + \left(\frac{pb}{a^2+b^2+c^2}\right)^2 + \left(\frac{pc}{a^2+b^2+c^2}\right)^2$$

$$= \frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2}{a^2+b^2+c^2}$$

3) A rectangular box open at the top is to have a volume of 32 cc. Find the dimension of the box, that require the least material for its construction.

Let the dimension of the box be x, y, z.

Volume $= xyz$.

Volume $= lbh$

Surface Area $= lb + 2h(l+$

$2(lb+bh+hl)$

Surface are $= xy + 2yz + 2zx$

Given volume $= 32$ cc $\Rightarrow xyz = 32$

(or) $xyz - 32 = 0$

$\therefore g = xyz - 32$

& $f = xy + 2yz + 2zx$.

$\therefore F(x,y,z) = xy + 2yz + 2zx + \lambda[xyz - 32]$

$\frac{\partial F}{\partial x} = y + 2z + \lambda yz \Rightarrow y + 2z + \lambda yz = 0 \quad \left[\because \frac{\partial F}{\partial x} = 0\right]$

$\qquad \longrightarrow (1)$

$\frac{\partial F}{\partial y} = x + 2z + \lambda xz$

$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2z + \lambda xz = 0 \quad \longrightarrow (2)$

$\frac{\partial F}{\partial z} = 2y + 2z + \lambda xy$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2y + 2x + \lambda xy = 0 \quad \longrightarrow (3)$

$① \times x \Rightarrow xy + 2xz + \lambda xyz = 0$

$② \times y \Rightarrow xy + 2yz + \lambda xyz = 0$

$$2xz - 2yz = 0$$

$$\Rightarrow x = y$$

Solve ② & ③:

$② \times y : \quad xy + 2yz + \lambda xyz = 0$

$③ \times z : \quad 2yz + 2xz + \lambda xyz = 0$

$$xy - 2zx = 0$$

$$\Rightarrow y = 2z.$$

$$\therefore x = y = 2z$$

Sub: $x = y = 2z$ in ①

$$x + 2\left(\frac{x}{2}\right) + \lambda(x)\left(\frac{x}{2}\right) = 0$$

$$x + x + \frac{\lambda x^2}{2} = 0$$

$$\Rightarrow \frac{\lambda x^2}{2} = -2x$$

$$\Rightarrow x = -4/\lambda$$

Sub $x = y = 2z$ in (2)

$$x + 2z + \lambda(xz) = 0$$

$$y + 2\frac{y}{2} + \lambda y \cdot \frac{y}{2} = 0$$

$$\Rightarrow y = -4/\lambda$$

Sub $x = y = 2z$ in (3)

$$2y + 2x + \lambda xy = 0$$

$$2(2z) + 2(2z) + \lambda(2z)(2z) = 0$$

$$4z + 4z + 4z^2\lambda = 0$$

$$2 + \cancel{4}z\lambda = 0$$

$$\Rightarrow z = -2/\lambda$$

$$\frac{\partial F}{\partial \lambda} = xyz - 32$$

$$\Rightarrow \frac{\partial F}{\partial \lambda} = 0 \Rightarrow xyz = 32$$

$$\Rightarrow -\frac{4}{\lambda} \times -\frac{4}{\lambda} \times -\frac{2}{\lambda} = 32$$

$$\Rightarrow -1 = \lambda \quad (\text{or}) \ \lambda = -1.$$

$\therefore x = 4, \ y = 4, \ z = 2$

$$\therefore f = xy + 2yz + 2zx.$$

$$= 4(4) + 2(4)(2) + 2(2)(4)$$

$$= 16 + 16 + 16 = 48.$$

3) Find the dimensions of the rectangular box without top of maxi. capacity with surface area 432 squ. metre.

Ans: $x = y = 2z$, 12, 12, 6. $V = 864$ cubic metres.