Development of Equations of Motion

The equations of motion for the airplane with controls locked can be written down in accordance with the Newtonian laws of motion which state that the summation of all external forces in any direction must equal the time rate of change of momentum, and the summation of all the moments of the external forces must equal the time rate of change of moment of momentum, all measured with respect to axes fixed in space.

By these laws the six equations of motion of the airplane with locked controls can be written down as follows:

$$\Sigma F_x = ma_x$$

$$\Sigma F_y = ma_y$$

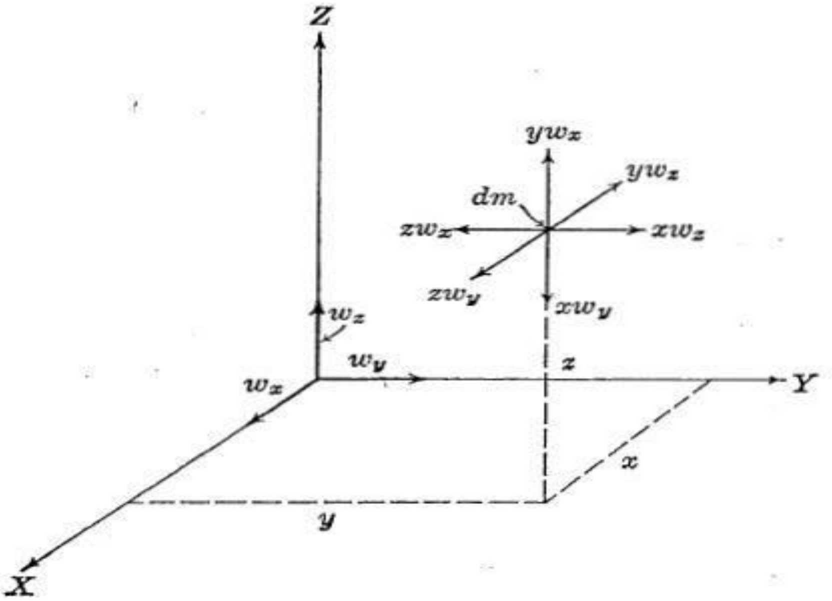
$$\Sigma F_z = ma_z$$

$$\Sigma L = \frac{dH_x}{dt}$$

$$\Sigma M = \frac{dH_y}{dt}$$

$$\Sigma N = \frac{dH_z}{dt}$$

where F_x, F_y, and F_z are the summation of the external forces; L, M, and N the moments of the external forces; and H_x, H_y, and H_z the moments of momentum along and about the fixed axes X, Y, and Z, respectively. As the acceleration and rates of change of moments of momentum must all be expressed along axes fixed in space, and the



axes chosen to represent the airplane (Figure 10–2) are moving axes, the acceleration and rates of change of moment of momentum along and about the airplane axes must be referred back to the axes fixed in space.

In order to determine the moment of momentum of a body about the X, Y, and Z axes, consider a body having an angular velocity, w, with components w_x, w_y, and w_z about OX, OY, and OZ, respectively.

The moment of momentum of dm about OX, OY, and OZ is as follows

$$dh_x = w_x(y^2 + z^2)\,dm - w_y xy\,dm - w_z zx\,dm$$
$$dh_y = w_y(z^2 + x^2)\,dm - w_z yz\,dm - w_x xy\,dm$$
$$dh_z = w_z(x^2 + y^2)\,dm - w_x zx\,dm - w_y yz\,dm$$

For the whole body, the moments of momentum about the three axes are integrals of equations (10–2):

$$h_x = w_x \int (y^2 + z^2)\, dm - w_y \int xy\, dm - w_z \int zx\, dm$$

$$h_y = w_y \int (z^2 + x^2)\, dm - w_z \int yz\, dm - w_x \int xy\, dm$$

$$h_z = w_z \int (x^2 + y^2)\, dm - w_x \int zx\, dm - w_y \int yz\, dm$$

The integral $\int (y^2 + z^2)\, dm$ is the moment of inertia of the body about the X axis, I_x, and the integral $\int xy\, dm$ is the product of inertia, J_{xy}. Equations (10–3) in terms of moments of inertia and products of inertia are as follows:

$$h_x = w_x I_x - w_y J_{xy} - w_z J_{xz}$$

$$h_y = w_y I_y - w_z J_{yz} - w_x J_{xy}$$

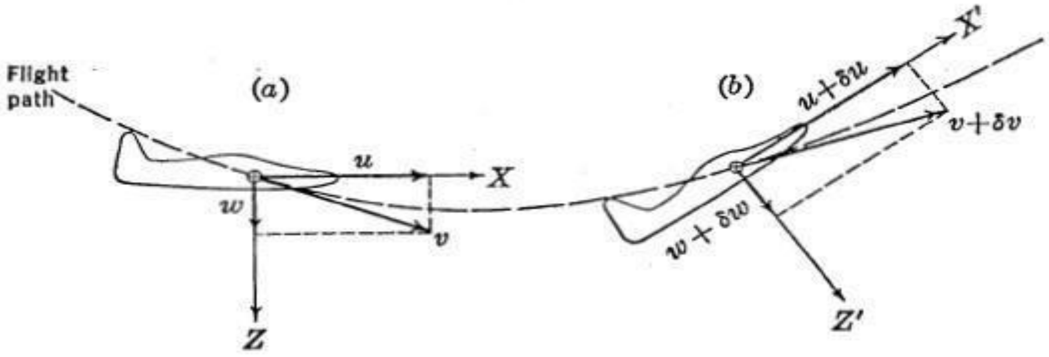
$$h_z = w_z I_z - w_x J_{xz} - w_y J_{yz}$$

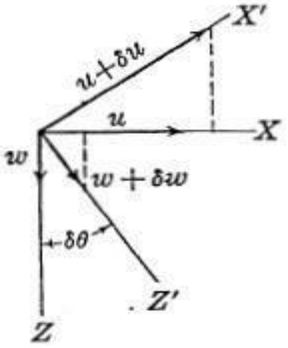
If the axes are chosen as principal axes, the products of inertia vanish. If the body contains a plane of symmetry, then the axis perpendicular to this plane will be a principal axis, thereby eliminating two of the products of inertia. The airplane is a body with a plane of symmetry taken as coinciding with the X-Z plane. The Y axis, perpendicular to this plane of symmetry, is therefore a principal axis, and the products of inertia, J_{xy} and J_{yz}, vanish. With the airplane motion terminology that the angular velocities w_x, w_y, and w_z are simply p, q, and r, and taking the Y axis as a principal axis, the moments of momentum of the airplane with respect to the X, Y, Z axes are as follows:

$$h_x = pI_x - rJ_{xz}$$

$$h_y = qI_y$$

$$h_z = rI_z - pJ_{xz}$$





The rate of change of moment of momentum relative to fixed space can be developed in a similar way from Figure 10–6.

$$\frac{dH_x}{dt} = \frac{dh_x}{dt} - h_y r + h_z q$$

$$\frac{dH_y}{dt} = \frac{dh_y}{dt} - h_z p + h_x r$$

$$\frac{dH_z}{dt} = \frac{dh_z}{dt} - h_x q + h_y p$$

In these expressions, a_x, a_y, a_z, dH_x/dt, dH_y/dt, dH_z/dt are all measured relative to fixed axes, whereas u, v, w, h_x, h_y, and h_z are measured relative to the moving axis. Equations (10–1) therefore may be written as follows, using equations (10–8) and (10–9):

$$\Sigma F_x = m(\dot{u} - vr + wq)$$

$$\Sigma F_y = m(\dot{v} - wp + ur)$$

$$\Sigma F_z = m(\dot{w} - uq + vp)$$

$$\Sigma L = (\dot{h}_x - h_y r + h_z q)$$

$$\Sigma M = (\dot{h}_y - h_z p + h_x r)$$

$$\Sigma N = (\dot{h}_z - h_x q + h_y p)$$

Making use of equations (10–5), the equations of motion of the airplane relative to the moving or Eulerian axes become:

$$\Sigma F_x = m(\dot{u} - vr + wq)$$

$$\Sigma F_y = m(\dot{v} - wp + ur)$$

$$\Sigma F_z = m(\dot{w} - uq + vp)$$

$$\Sigma L = \dot{p}I_x - \dot{r}J_{xz} + (I_z - I_y)qr - pqJ_{xz}$$

$$\Sigma M = \dot{q}I_y + rp(I_x - I_z) + (p^2 - r^2)J_{xz}$$

$$\Sigma N = \dot{r}I_z - \dot{p}J_{xz} + (I_y - I_x)pq + J_{xz}qr$$