

## Bending of beams

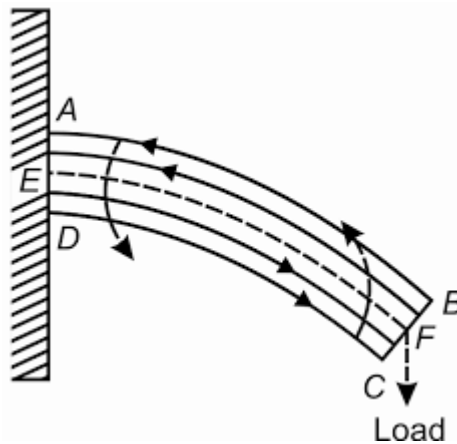
A beam is defined as a rod (or) bar of uniform cross section whose length is very much greater than its other dimensions, such as breadth and thickness.

It is commonly used in the construction of bridges to support roofs of the buildings etc. Since the length of the beam is much greater than its other dimensions the shearing stresses are very small.

### Assumptions:

- The length of the beam should be large compared to other dimensions.
- The applied load should be large compared to the weight of the beam.
- The cross section of the beam remains constant and hence the geometrical moment of inertia also remains constant.
- The shearing stresses are negligible.

Consider a beam ABCD, which is made up of a large number of thin plane layers are placed one above the other. One end of the beam AD is fixed in the rigid support and a Load is applied in the other end BC.



Taking the longitudinal section ABCD of the bent beam, the layers in the upper half are elongated while those in the lower half are compressed.

In the middle there is a layer (MN) which is not elongated or compressed due to bending of the beam. This layer is called the '**neutral surface**' and the line (MN) at which the neutral layer intersects the plane of bending is called the '**neutral axis**'.

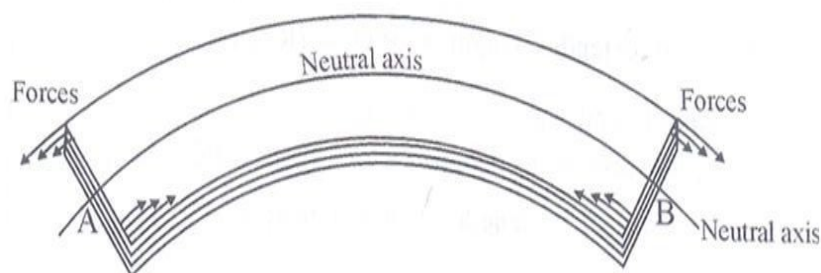
The layers below MN are compressed and those above MN are elongated and there will be such pairs of layers one above MN and one below MN experiencing same forces of elongation and compression due to bending and each pair forms a couple.

*The resultant moments of all these internal couples are called the **internal bending moment** and in the equilibrium condition, this is equal to the external bending moment*  
**INTERNAL BENDING MOMENT OF THE BEAM**

When a beam bent, the restoring couple arises. This couple balances the external couple due to external load is called **internal bending moment of the beam**.  
At equilibrium,

**Restoring couple = Bending couple**

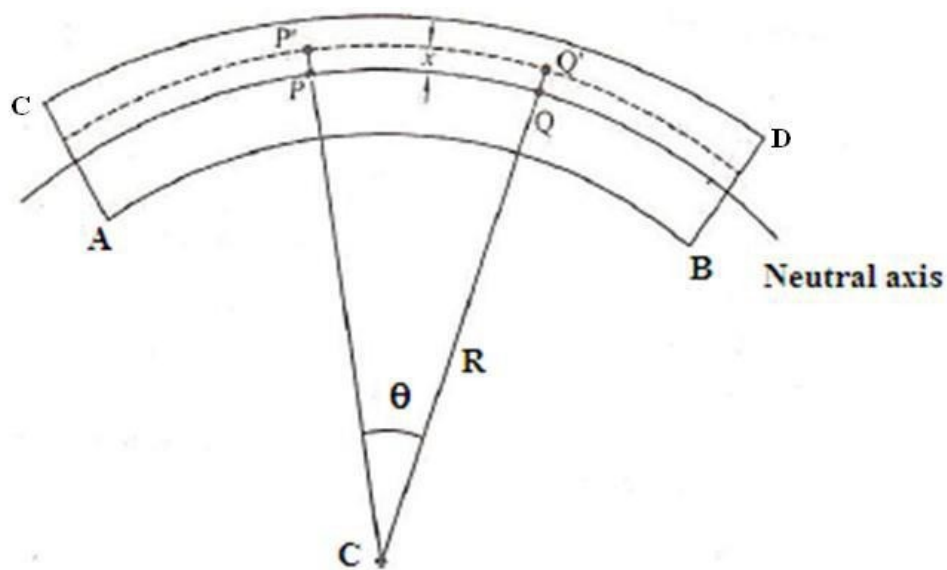
A beam may be assumed to consist of a number of parallel longitudinal metallic fibers placed one over the other and are called as filaments. Let the beam be subjected to deforming forces at its ends, due to which it bends. Let us consider a filament AB at the center of the beam. It is found that the filaments (layers) lying above AB gets elongated, while the filaments lying below AB gets compressed.



Bending of beam.



Therefore the filament i.e. layer AB which remains unaltered taken as the reference axis called as Neutral axis and the plane is called as neutral plane. Further, the deformation of any filament can be measured with reference to the neutral axis. The moment of couple due to elastic reactions (restoring couple) which balances the bending couple due to applied load is called the bending moment. Let us consider a beam under the action of deforming forces. The beam bends into a circular arc.



Let PQ be the neutral axis of the beam and P' Q' be another filament at distance y from PQ. If R is the radius of curvature of the neutral axis and  $\theta$  is the angle subtended by it at its centre of curvature 'C'.

Then we can write original length  $PQ = \text{Radius} \times \text{Angle}$

$$= R\theta \dots \dots \dots (1)$$

If  $R \times$  is the radius of curvature of the filament P'Q'.

$$P'Q' = (R \times)\theta \dots \dots \dots (2)$$

Extension Produced in the filament P'Q' due to bending = P'Q' - PQ

$$= (R \times)\theta - R\theta$$

$$= x\theta \dots \dots \dots (3)$$



Longitudinal strain=

$$\frac{\text{Change in length}}{\text{Original length}} \times \frac{x}{R}$$



The Young's modulus of the filament P'Q'

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

Longitudinal stress on the filament P'Q' = Y longitudinal strain

$$= Y \frac{x}{R}$$

R

If A is the area of cross-section of the filament, then the tensile force on the filament.

$$= \text{Longitudinal stress} \times \text{area}$$

$$= Y \frac{x}{R} A$$

We know, moment of longitudinal force about the neutral axis = tensile force × distance

$$Y \frac{x}{R} A \times \frac{Y}{R} = \frac{Y^2}{R^2} A x^2$$

$$\text{Moment of all the forces about the neutral axis} = \frac{Y}{R} \frac{Y}{R} A x^2 = \frac{Y^2}{R^2} I$$

$I_g$  is the geometrical moment of inertia and it is represented as  $AK^2$

Where A is the area of the cross-section and K is radius of gyration. In equilibrium,

Bending moment of the beam = Moment of force

$$\text{Internal bending moment of the beam} = YI \frac{4}{R^g}$$

## SPECIAL CASES

### Rectangular cross section

If b and d are the breadth and thickness of the beam, then

$$A = bd \text{ and } K^2 = \frac{d^2}{12}$$

$$I_g = AK^2 = \frac{bd^3}{12}$$

Using the value of  $I_g = \frac{bd^3}{12}$  in equation (4)



|   |  |
|---|--|
| □ Internal bending moment of the rectangular beam | $\frac{bd^3Y}{12R} \dots\dots\dots(5)$ |
|---|--|



### Circular crosssection

If  $r$  be the radius of the beam, then  $A = \pi r^2$  and  $K = \frac{\pi r^4}{4}$

$$I_g = \frac{\pi r^4}{4}$$

Using the value  $I_g = \frac{\pi r^4}{4}$  in equation (4)

$$\text{Internal bending moment of the circular beam} = \frac{r^4 Y}{4R} \dots\dots\dots(6)$$

### DEPRESSION OF A CANTILEVER

#### DEFINITION

A light beam clamped horizontally at one end and loaded with a weight  $W = Mg$  at the free end is called a **cantilever**.

In equilibrium,

$$\text{External bending moment} = \text{Internal bending moment}$$

#### THEORY

Let us consider a beam fixed at one end and loaded at its other free end as shown in fig 1.7.2.2. Let AB is the neutral axis of a cantilever (a beam or rod) of length 'l' is fixed at the end A and loaded at the free end B by a weight W. Due to load applied the cantilever is depressed to B'.

Let BB' represents the vertical depression at the free end.



Due to the load applied at the free end, a couple is created between the two forces. (i.e)

(i) Force (load 'W') applied at the free end towards downward direction and





(ii) Reaction (R) acting in the upward direction at the supporting end.

This external bending couple tends to bend the beam in the clockwise direction. But, since one end of the beam is fixed, the beam cannot rotate. Therefore the external bending couple must be balanced by another equal and opposite couple, created due to the elastic nature of the body called as internal bending moment.

Consider the section of the cantilever P at a distance 'x' from the fixed end A. Q is another point at a distance dx from P i.e., PQ = dx. It is at a distance (l-x) from the loaded end B'. Considering the equilibrium of the portion PB', there is a force of reaction W of P. Let O be the centre of curvature and R be the radius of curvature.

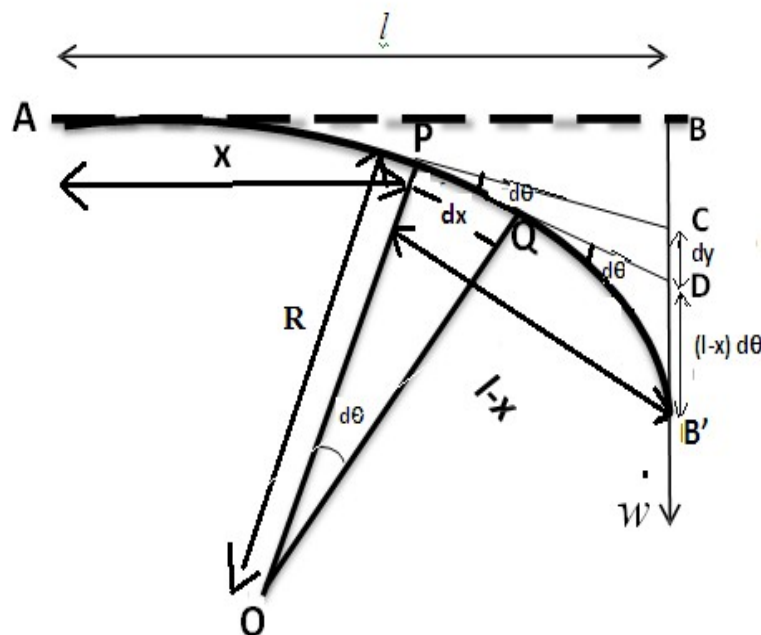


Fig1.7.2.2. Depression of a cantilever.

• The external bending moment =  $W PB' = W(l-x)$ .....(1)

• Internal bending moment of the cantilever  $\frac{Y}{R} L$  .....(2)



R-Radius of the curvature of the neutral axis at P.

Where Y – Young’s modulus of the cantilever.

$I_g$  - Geometrical moment of inertia of its cross-section.

In the equilibrium position,

External bending moment = Internal bending moment

$$R^g \frac{W(l-x) Y I}{R} = \dots\dots\dots(3)$$

Let Q be another point on the bent cantilever at small distance ‘dx’ from P. Since P and Q are very near, we can assume that the radius of curvature R is practically the same.

The tangents are drawn at P and Q meeting the vertical line BB’ at C and D. Let dθ be the angle between the tangents at P and Q.

$$\sin \frac{d\theta}{R} = \frac{dx}{d} \dots\dots\dots(4)$$

Substituting the value of R from(3) in(4), we have

$$\frac{d\theta}{Y I_g} = \frac{W(l-x) dx}{d} \dots\dots\dots(5)$$

If ‘dy’ is the depression due to the curvature at PQ

$$dy = \int (l-x) d\theta \dots\dots\dots(6)$$

Substituting value of dθ

$$dy = \int (l-x) \frac{W(l-x) dx}{Y I_g} \dots\dots\dots(7)$$



R-Radius of the curvature of the neutral axis at P.

To find the total depression at the free end of the cantilever equation (7) has to be integrated from 0 to  $l$ .

$$dy = \frac{W}{YI_g} \int_0^l (l-x)^2 dx$$



$$\frac{y}{YI_g} = \frac{W}{2} \int_0^l (lx)^2 dx = \frac{W}{YI_g} \int_0^l (l^2 x^2) dx$$

$$= \frac{W}{YI_g} \left[ \frac{l^2 x^3}{3} \right]_0^l = \frac{W}{YI_g} \left[ \frac{l^3}{3} - 0 \right]$$

.....(8)

Equation(8)gives

$$y = \frac{Wl^3}{3YI_g}$$

produced by the cantilever at the free end.

**SPECIAL CASES**

**Rectangular crosssection**

If b and d are the breadth and thickness of the beam, then

$$A = bd \text{ and } K = \frac{bd^3}{12}$$

Substituting the values of

$$I_g = \frac{AK^2}{4} = \frac{bd^3}{12} \text{ and } W = Mg \text{ in equation (8)}$$

$$\text{Depression } y = \frac{4Mgl^3}{3Ybd^3} \text{ .....(9)}$$

**Circular crosssection**

If r be the radius of the beam, then

$$A = r^2 \text{ and } K = \frac{r^4}{4}$$

Substituting the values of

$$I_g = \frac{AK^2}{4} = \frac{r^4}{4} \text{ and } W = Mg \text{ in equation (8)}$$

$$\text{Depression } y = \frac{4Mgl^3}{3Yr^4} \text{ .....(10)}$$

**EXPERIMENTAL DETERMINATION OF YOUNG'S MODULUS BY CANTILEVER DEPRESSION METHOD**

**CONSTRUCTION**



One end of the beam is rigidly clamped at one end to the edge of the table using G-clamp. A tall pin P is fixed vertically to the free end of the bar. A loop of cotton string or a hook is attached to this end of the bar and a weight hanger is



suspended from it. A travelling microscope is focused on the tip of the pin as shown in fig.

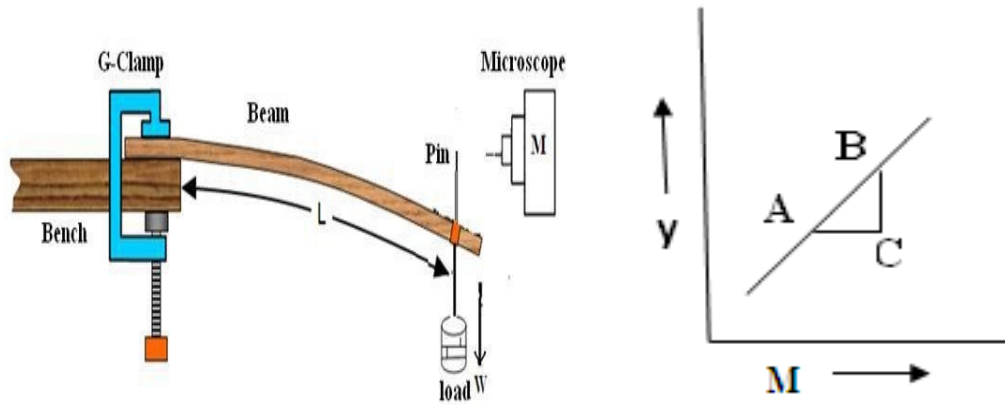


Fig1.7.2.3 Experimental verification of a cantilever.

## PROCEDURE

A dead load without any slotted weights is attached to the hook. The microscope is adjusted such that the horizontal cross wire coincides with the tip of the image of the pin and the reading on the vertical scale is taken. Loads are added to the hanger in steps of 50g and every time, the readings are noted on the vertical scale. A travelling microscope is focused on the tip of the pin as shown in fig. These observations are also repeated while unloading in the same. Steps and the readings are tabulated. The mean depression 'y' for a load 'M' kg is found from the tabulated readings.

The observations are tabulated as follows.

## Graphical method

A graph is drawn between the load (M) along X-axis and elevation (y) along Y-axis. It is found to be a straight line as shown in fig. The slope of the straight line gives the value of  $(y/M)$ .



| Load<br>$10^3 kg$ | Microscopereadings |           |          |           |           |          |                  | depression<br>(y)for100g<br>$10^2 m$ | Mean<br>(M/y)<br>$Kgm^{-1}$ |
|-------------------|--------------------|-----------|----------|-----------|-----------|----------|------------------|--------------------------------------|-----------------------------|
|                   | Loading            |           |          | Unloading |           |          | Mean<br>$10^2 m$ |                                      |                             |
|                   | MSRc<br>m          | VSR<br>cm | TR<br>cm | MSRc<br>m | VSR<br>Cm | TR<br>Cm |                  |                                      |                             |
| W                 |                    |           |          |           |           |          |                  |                                      |                             |
| W+50              |                    |           |          |           |           |          |                  |                                      |                             |
| W+100             |                    |           |          |           |           |          |                  |                                      |                             |
| W+150             |                    |           |          |           |           |          |                  |                                      |                             |
| W+200             |                    |           |          |           |           |          |                  |                                      |                             |

Hence Young's modulus of the cantilever can be calculated as

$$Y = \frac{4gl^3 M}{bd^3} \frac{1}{y}$$