$\Rightarrow$ Cayley Hamilton theorem

Period:

Every square matrix satisfies its own characteristic equation.

$\Rightarrow$ Uses of Cayley hamilton theorem:-

1) positive integrals power of A

2) $A^{-1} \Rightarrow$ Inverse of A

1) Verify cayley hamilton theorem for

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$ Find $A^4$ and $A^{-1}$

Soln:- The characteristic equation

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

$D_1 = 2+2+2 = 6$

$D_2 = (4-1) + (4-1) + (4-1)$

$= 3+3+3 = 9$

$D_3 = 2(4-1) - (-1)(-2+1) + 1(1-2)$

$= 2(3) + 1(-1) + 1(-1)$

$= 6-1-1 = 4$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By cayley hamilton theorem,

$A^3 - 6A^2 + 9A - 4 = 0$

$\hookrightarrow A^3 - 6A^2 + 9A - 4I = 0$

To find:- $A^2, A^3$

Now

$$A^2 = A \times A \quad \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -6 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

Now

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$\begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6 \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$- 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - \begin{pmatrix} 36 & -30 & +30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{pmatrix} + \begin{pmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{pmatrix}$$

$$- \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 0$$

$$= \begin{pmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

To find $A^4$

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$A^3 = 6A^2 - 9A + 4I \qquad (\because IA = A)$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

$$A^4 = 6\begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 9\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 4\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 132 & -126 & 126 \\ -126 & 132 & -126 \\ 126 & -126 & 132 \end{pmatrix} - \begin{pmatrix} 54 & -45 & 45 \\ -45 & 54 & -45 \\ 45 & -45 & 54 \end{pmatrix} + \begin{pmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 132 - 54 + 8 & -126 + 45 - 4 & 126 - 45 + 4 \\ -126 + 45 - 4 & 132 - 54 + 8 & -126 + 45 - 4 \\ 126 - 45 + 4 & -126 + 45 - 4 & 132 - 54 + 8 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{pmatrix}$$

To find $A^{-1}$

$$A^3 - 6A^2 + 9A - 4I = 0 \qquad \begin{pmatrix} \because A^{-1}A = I \\ I A^{-1} = A^{-1} \end{pmatrix}$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}\left[ A^2 - 6A + 9I \right]$$

$$A^{-1} = \frac{1}{4}\left[ \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$A^{-1} = \frac{1}{4}\left[\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 6-12+9 & -5+6+0 & 15-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{pmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 3/4 \end{pmatrix}$$

13.10.23 Period:3

2, Verify cayley hamilton theorem for $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$, Find $A^4$ and $A^{-1}$

Soln:- The characteristic equation

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

$D_1 = 1+1+1 = 3$

$D_2 = (1-1) + (1-3) + (1-0)$

$= 0-2+1 = -1$

$D_3 = 1(1-1) - 0 + 3(-2-1)$

$= 3(-3) = -9$

$\therefore \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By cayley Hamilton theorem

$A^3 - 3A^2 - A + 9 = 0$

$\hookrightarrow A^3 - 3A^2 - A + 9I = 0$

To find :- $A^2, A^3$

Now

$A^2 = A \times A$

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

$A^3 = A^2 \times A$

$$= \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix}$$

Now :-

$A^3 - 3A^2 - A + 9I = 0$

$$\begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} - 3\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} - \begin{pmatrix} 12 & -9 & 18 \\ 9 & 6 & 12 \\ 0 & -6 & 15 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 4-12-1+9 & -9+9-0+0 & 21-18-3+0 \\ 11-9-2+0 & -2-6-1+9 & 11-12+1+0 \\ 1-0-1+0 & -7+6+1+0 & 7-15-1+9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

To find $A^4, A^{-1}$

$$A^3 - 3A^2 - A + 9I = 0$$

$$A^4 = 3A^3 + A^2 - 9A$$

$$A^4 = 3\begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - 9\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & -27 & 21 \\ 33 & -6 & 33 \\ 3 & -21 & 21 \end{pmatrix} + \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} 9 & 0 & 27 \\ 18 & 9 & -9 \\ 9 & -9 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 12+4-9 & -27-3-0 & 21+6-27 \\ 33+3-18 & -6+2-9 & 33+4+9 \\ 3+0-9 & -21-2+9 & 21+5-9 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -30 & 0 \\ 18 & -1 & 46 \\ -6 & -14 & 17 \end{pmatrix}$$

To find :- $A^{-1}$

$$A^2 - 3A - I + 9A^{-1} = 0$$

$$A^{-1} = [-A^2 + 3A + I]\, 1/9$$

$$A^{-1} = \left[ -\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + 3\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

$$A^{-1} = \left[\begin{pmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 3 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$A^{-1} = \begin{bmatrix} -4+3+1 & 3+0+0 & -6+3+0 \\ -3+6+0 & -2+3+1 & -4-3+0 \\ 0+3+0 & -2-3+0 & -5+3+1 \end{bmatrix}$$

$$A^{-1} = \begin{pmatrix} 0 & 3 & -3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix} \frac{1}{9}$$

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 0 & 3 & -3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix}$$

3, verify cayley hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, find $A^3$ and $A^{-1}$.

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation is

$$\lambda^2 - D_1\lambda + D_2 = 0$$

$D_1 = 1+3 = 4$

$D_2 = |A|$

$= \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3-8 = -5$

$\therefore \lambda^2 - 4\lambda - 5 = 0$

By cayley hamilton theorem

$$A^2 - 4A - 5I = 0$$

$A^2 = A \times A$

$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8=0 & 17-12-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

To find $A^3$, $A^{-1}$

$$A^2 - 4A - 5I = 0$$

$$A^3 = 4A^2 - 5A$$

$$A^3 = 4\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 5\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 64 \\ 32 & 68 \end{bmatrix} - \begin{bmatrix} 5 & 20 \\ 10 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 36-5 & 64-20 \\ 32-10 & 68-15 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 44 \\ 22 & 53 \end{bmatrix}$$

$$A^2 - 4A - 5I = 0$$

$$A - 4I - 5A^{-1} = 0$$

$$A^{-1} = 1/5\,[A - 4I]$$

$$A^{-1} = \frac{1}{5}\left[\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{5}\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{5}\left[\begin{pmatrix} 1-4 & 4-0 \\ 2-0 & 3-4 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

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4) Use Cayley hamilton theorem, to find the value of $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = 0$ where $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ Period: 6

The characteristic equation

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

$D_1$ = sum of the main diagonal elements.

$= 2 + 1 + 2$

$\boxed{D_1 = 5}$

$D_2$ = sum of the minors of main diagonal elements.

$= (2-0) + (4-1) + (2-0)$

$= 2 + 3 + 2 = 7$

$D_3 = |A|$

$= 2(2-0) - 1(0) + 1(0-1)$

$= 2(2) + 1(-1)$

$= 4 - 1 = 3$

$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

using cayley hamilton theorem:-

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A + 8A^2 - 2A + I = 0$$

$$\begin{array}{l|l}
 & A^5 + A \\
\hline
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\
 & A^8 - 5A^7 + 7A^6 - 3A^5 \\
\hline
 & A^4 - 5A^3 + 8A^2 - 2A \\
 & A^4 - 5A^3 + 7A^2 - 3A \\
\hline
 & A^2 + A + I
\end{array}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$

$$= 0 + (A^2 + A + I)$$

$$= A^2 + A + I$$

$$A^2 = A \times A$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 0 \end{pmatrix}$$

$$A^2 + A + I = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

5) Use Hayley hamilton theorem. $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. To express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ As linear polynomial in A. And find the value

The characteristic equation is

$$\lambda^2 - D_1\lambda + D_2 = 0$$

$$D_1 = 1+3 = 4$$

$$D_2 = 3-8 = -5$$

$$\therefore \lambda^2 - 4\lambda - 5 = 0 \Rightarrow A^2 - 4A - 5 = 0$$

$$\Rightarrow A^2 - 4A - 5I =$$

using cayley Hamilton theorem,

$$A^2 - 4A - 5I$$

$$\begin{array}{r|l} & A^3 - 2A + 3 \\ \hline A^2 - 4A - 5I & A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I \\ & A^5 - 4A^4 - 5A^3 \\ \hline & -2A^3 + 11A^2 - A \\ & -2A^3 + 8A^2 + 10A \\ & (+) \quad (-) \quad (-) \\ \hline & 3A^2 - 11A - 10I \\ & 3A^2 - 12A - 15I \\ & (+) \quad (+) \quad (+) \\ \hline & A + 5I \end{array}$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^3 - 2A + 3)(A^2 - 4A - 5I) + (A + 5I)$$

$$= (A + 5I)$$

$$= 0 + (A + 5I)$$

$$= (A + 5I)$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$$