Columns and Struts

19.1. INTRODUCTION

Column or strut is defined as a member of a structure, which is subjected to axial compressive load. If the member of the structure is vertical and both of its ends are fixed rigidly while subjected to axial compressive load, the member is known as *column*, for example a vertical pillar between the roof and floor. If the member of the structure is not vertical and one or both of its ends are hinged or pin joined, the bar is known as *strut*. Examples of struts are : connecting rods, piston rods etc.

19.2. FAILURE OF A COLUMN

The failure of a column takes place due to the anyone of the following stresses set up in the columns :

(*i*) Direct compressive stresses,

(*ii*) Buckling stresses, and

(*iii*) Combined of direct compressive and buckling stresses.

19.2.1. Failure of a Short Column. A short column of uniform cross-sectional area A, subjected to an axial compressive load P, is shown in Fig. 19.1. The compressive stress induced is given by

$$p = \frac{P}{A}$$

If the compressive load on the short column is gradually increased, a stage will reach when the column will be on the point of failure by crushing. The stress induced in the column corresponding to this load is known as crushing stress and the load is called crushing load.

Let P_c = Crushing load,

σ_c = Crushing stress, and

A = Area of cross-section.

Then $$\sigma_c = \frac{P_c}{A}.$$

All short columns fail due to crushing.

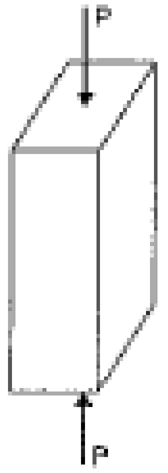

Fig. 19.1

19.2.2. Failure of a Long Column. A long column of uniform cross-sectional area A and of length l, subjected to an axial compressive load P, is shown in Fig. 19.2. A column is known as long column if the length of the column in comparison to its lateral dimensions, is very large. Such columns do not fail by crushing alone, but also by bending (also known buckling) as shown in Fig. 19.2. The load at which the column just buckles, is known as *buckling load or critical just or crippling load*. The buckling load is less than the crushing load for a long column. Actually the value of buckling load for long columns is low whereas for short columns the value of buckling load is relatively high.

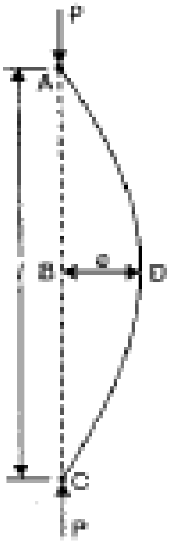

Fig. 19.2

Refer to Fig. 19.2.

Let l = Length of a long column

P = Load (compressive) at which the column has just buckled

A = Cross-sectional area of the column

e = Maximum bending of the column at the centre

σ_0 = Stress due to direct load = $\frac{P}{A}$

σ_b = Stress due to bending at the centre of the column = $\frac{P \times e}{Z}$

where Z = Section modulus about the axis of bending.

The extreme stresses on the mid-section are given by

Maximum stress = $\sigma_0 + \sigma_b$

and Minimum stress = $\sigma_0 - \sigma_b$.

The column will fail when maximum stress (*i.e.*, $\sigma_0 + \sigma_b$) is more than the crushing stress σ_c. But in case of long columns, the direct compressive stresses are negligible as compared to buckling stresses. Hence very long columns are subjected to buckling stresses only.

19.3. ASSUMPTIONS MADE IN THE EULER'S COLUMN THEORY

The following assumptions are made in the Euler's column theory :

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

19.4. END CONDITIONS FOR LONG COLUMNS

In case of long columns, the stress due to direct load is very small in comparison with the stress due to buckling. Hence the failure of long columns take place entirely due to buckling (or bending). The following four types of end conditions of the columns are important :

1. Both the ends of the column are hingrd (or pinned).
2. One end is fixed and the other end is free.
3. Both the ends of the column are fixed.
4. One end is fixed and the other is pinned.

For a hinged end, the deflection is zero. For a fixed end the deflection and slope are zero. For a free end the deflection is not zero.

19.4.1. Sign Conventions. The following sign conventions for the bending of the columns will be used :

1. A moment which will bend the column with its *convexity* towards its initial central line as shown in Fig. 19.3 (*a*) is taken as positive. In Fig. 19.3 (*a*), *AB* represents the initial centre line of a column. Whether the column bends taking the shape *AB'* or *AB"*, the moment producing this type of curvature is positive.

2. A moment which will tend to bend the column with its *concavity* towards its initial centre line as shown in Fig. 19.3 (*b*) is taken as negative.

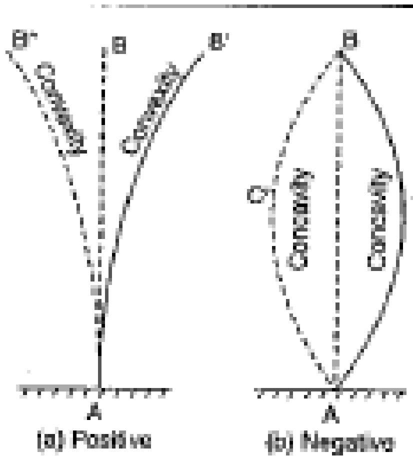

Fig. 19.3

19.5. EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE HINGED

The load at which the column just buckles (or bends) is called crippling load. Consider a column *AB* of length *l* and uniform cross-sectional area, hinged at both of its ends *A* and *B*. Let *P* be the crippling load at which the column has just buckled. Due to the crippling load, the column will deflect into a curved form *ACB* as shown in Fig. 19.4.

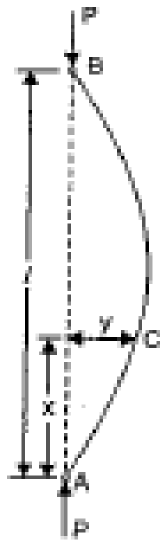

Fig. 19.4

Consider any section at a distance x from the end *A*.

Let y = Deflection (lateral displacement) at the section.

The moment due to the crippling load at the section $= -P \cdot y$

(– ve sign is taken due to sign convention given in Art. 19.4.1)

But moment $= EI \dfrac{d^2y}{dx^2}$.

Equating the two moments, we have

$$EI \frac{d^2y}{dx^2} = -P \cdot y \quad \text{or} \quad EI \frac{d^2y}{dx^2} + P \cdot y = 0$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right) \qquad \ldots(i)$$

where C_1 and C_2 are the constants of integration. The values of C_1 and C_2 are obtained as given below :

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*The equation $\dfrac{d^2y}{dx^2} + \dfrac{P}{EI} \times y = 0$ can be written as $\dfrac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \dfrac{P}{EI}$ or $\alpha = \sqrt{\dfrac{P}{EI}}$

The solution of the equation is $y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) \text{ as } \alpha = \sqrt{\frac{P}{EI}}$$

(*i*) At A, $x = 0$ and $y = 0$ (See Fig. 19.4)

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos 0^\circ + C_2 \sin 0$$
$$= C_1 \times 1 + C_2 \times 0 \qquad (\because \cos 0 = 1 \text{ and } \sin 0 = 0)$$
$$= C_1$$

$\therefore$ $$C_1 = 0. \qquad ...(ii)$$

(*ii*) At B, $x = l$ and $y = 0$ (See Fig. 19.4).

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos\left(l \times \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right)$$
$$= 0 + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right) \qquad [\because C_1 = 0 \text{ from equation } (ii)]$$
$$= C_2 \sin\left(l\sqrt{\frac{P}{EI}}\right) \qquad ...(iii)$$

From equation (*iii*), it is clear that either $C_2 = 0$

or $$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0.$$

As $C_1 = 0$, then if C_2 is also equal to zero, then from equation (*i*) we will get $y = 0$. This means that the bending of the column will be zero or the column will not bend at all. Which is not true.

$\therefore$ $$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$
$$= \sin 0 \text{ or } \sin \pi \text{ or } \sin 2\pi \text{ or } \sin 3\pi \text{ or } ...$$

or $$l\sqrt{\frac{P}{EI}} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } ...$$

Taking the least practical value,

$$l\sqrt{\frac{P}{EI}} = \pi$$

or $$P = \frac{\pi^2 EI}{l^2}. \qquad ...(19.1)$$

19.6. EXPRESSION FOR CRIPPLING LOAD WHEN ONE END OF THE COLUMN IS FIXED AND THE OTHER END IS FREE

Consider a column AB, of length l and uniform cross-sectional area, fixed at the end A and free at the end B. The free end will sway sideways when load is applied at free end and curvature in the length l will be similar to that of upper half of the column whose both ends are hinged. Let P is the crippling load at which the column has just buckled. Due to the crippling load P, the column will deflect as shown in Fig. 19.5 in which AB is the original position of the column and AB', is the deflected position due to crippling load P.

Consider any section at a distance x from the fixed end A.

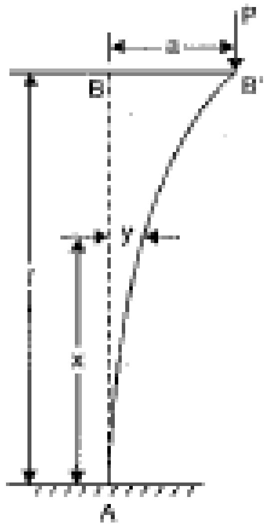

Fig. 19.5

Let y = Deflection (or lateral displacement) at the section

a = Deflection at the free end B.

Then moment at the section due to the crippling load = $P\,(a - y)$

(+ve sign is taken due to sign convention given in Art. 19.4.1)

But moment is also $= EI\,\dfrac{d^2y}{dx^2}$

∴ Equating the two moments, we get

$$EI\,\frac{d^2y}{dx^2} = P\,(a - y) = P\,.\,a - P\,.\,y$$

or
$$EI\,\frac{d^2y}{dx^2} + P\,.\,y = P\,.\,a$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI}\,.\,y = \frac{P}{EI}\,.\,a. \qquad \text{...(A)}$$

The solution* of the differential equation is

$$y = C_1\,.\,\cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2\,.\,\sin\left(x\sqrt{\frac{P}{EI}}\right) + a \qquad \text{...(i)}$$

where C_1 and C_2 are constant of integration. The values of C_1 and C_2 are obtained from boundary conditions. The boundary conditions are :

(*i*) For a fixed end, the deflection as well as slope is zero.

Hence at end A (which is fixed), the deflection $y = 0$ and also slope $\dfrac{dy}{dx} = 0$.

Hence at A, $x = 0$ and $y = 0$

Substituting these values in equation (*i*), we get

$$0 = C_1\,.\,\cos 0 + C_2 \sin 0 + a$$
$$= C_1 \times 1 + C_2 \times 0 + a \qquad (\because \cos 0 = 1, \sin 0 = 0)$$
$$= C_1 + a$$

∴
$$C_1 = -\,a \qquad \text{...(ii)}$$

At A, $x = 0$ and $\dfrac{dy}{dx} = 0$.

Differentiating the equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = C_1\,.\,(-1)\sin\left(x\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + 0$$

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*The equation (A) can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 \times y = \alpha^2 \times a \text{ where } \alpha^2 = \frac{P}{EI} \text{ or } \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of this equation is $y = C_1 \cos(\alpha\,.\,x) + C_2 \sin(\alpha\,.\,x) + \dfrac{\alpha^2 \times a}{\alpha^2}$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + a$$

$$= -C_1 \cdot \sqrt{\frac{P}{EI}} \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sqrt{\frac{P}{EI}} \cos\left(x\sqrt{\frac{P}{EI}}\right)$$

But at A, $x = 0$ and $\frac{dy}{dx} = 0$.

∴ The above equation becomes as

$$0 = -C_1 \cdot \sqrt{\frac{P}{EI}} \sin 0 + C_2 \sqrt{\frac{P}{EI}} \cos 0$$

$$= -C_1 \sqrt{\frac{P}{EI}} \times 0 + C_2 \cdot \sqrt{\frac{P}{EI}} \times 1 = C_2 \sqrt{\frac{P}{EI}}.$$

From the above equation it is clear that either $C_2 = 0$.

or $$\sqrt{\frac{P}{EI}} = 0.$$

But for the crippling load P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

∴ $$C_2 = 0.$$

Substituting the values of $C_1 = -a$ and $C_2 = 0$ in equation (*i*), we get

$$y = -a \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + a. \quad \text{...(iii)}$$

But at the free end of the column, $x = l$ and $y = a$.

Substituting these values in equation (*iii*), we get

$$a = -a \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) + a$$

or $$0 = -a \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) \text{ or } a \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = 0$$

But 'a' cannot be equal to zero

∴ $$\cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = 0 = \cos\frac{\pi}{2} \text{ or } \cos\frac{3\pi}{2} \text{ or } \cos\frac{5\pi}{2} \text{ or ......}$$

∴ $$l \cdot \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ .....}$$

Taking the least practical value,

$$l \cdot \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \quad \text{or} \quad \sqrt{\frac{P}{EI}} = \frac{\pi}{2l}$$

or $$P = \frac{\pi^2 EI}{4l^2} \quad \text{...(19.2)}$$

19.7. EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE FIXED

Consider a column AB of length l and uniform cross-sectional area fixed at both of its ends A and B as shown in Fig. 19.6. Let P is the crippling load at which the column has buckled.

Due to the crippling load P, the column will deflect as shown in Fig. 19.6. Due to fixed ends, there will be fixed end moments (say M_0) at the ends A and B. The fixed end moments will be acting in such direction so that slope at the fixed ends becomes zero.

Consider a section at a distance x from the end A. Let the deflection of the column at the section is y. As both the ends of the column are fixed and the column carries a crippling load, there will be some fixed end moments at A and B.

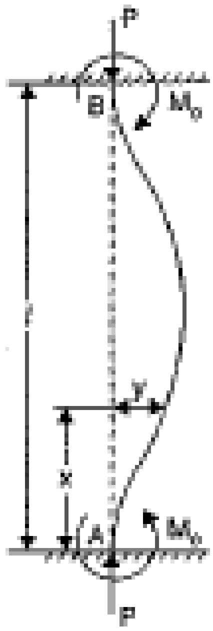

Fig. 19.6

Let M_0 = Fixed end moments at A and B.

Then moment at the section = $M_0 - P.y$

But moment at the section is also = $EI \frac{d^2y}{dx^2}$

$\therefore$ Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = M_0 - P.y$$

or
$$EI \frac{d^2y}{dx^2} + P \cdot y = M_0$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{M_0}{EI} \qquad ...(A)$$

$$= \frac{M_0}{EI} \times \frac{P}{P} = \frac{P}{EI} \cdot \frac{M_0}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad ...(i)$$

where C_1 and C_2 are constant of integration and their values are obtained from boundary conditions. Boundary conditions are :

(i) At A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$ as A is a fixed end.

(ii) At B, $x = l$, $y = 0$ and also $\frac{dy}{dx} = 0$ as B is also a fixed end.

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*The equation (A) can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 \times y = \frac{M_0}{EI} \quad \text{where } \alpha^2 = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of this equation is

$$y = C_1 \cos(\alpha \cdot x) + C_2 \sin(\alpha \cdot x) + \frac{M_0}{EI \times \alpha^2}$$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + \frac{M_0}{EI \times \frac{P}{EI}}$$

$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \frac{P}{\sqrt{EI}}\right) + \frac{M_0}{P}$$

Substituting the value $x = 0$ and $y = 0$ in equation (*i*), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{M_0}{P} \qquad (\because \quad \cos 0 = 1)$$

$$= C_1 + \frac{M_0}{P}$$

$$\therefore \qquad C_1 = -\frac{M_0}{P} \qquad ...(ii)$$

Differentiating the equation (*i*), with respect to x, we get

$$\frac{dy}{dx} = C_1 \,.\, (-1) \sin\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}} + 0$$

$$= -C_1 \sin\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}}$$

Substituting the value $x = 0$ and $\frac{dy}{dx} = 0$, the above equation becomes

$$0 = -C_1 \times 0 + C_2 \times 1 \times \sqrt{\frac{P}{EI}} \qquad (\because \quad \sin 0 = 0 \text{ and } \cos 0 = 1)$$

$$= C_2 \sqrt{\frac{P}{EI}} \,.$$

From the above equation, it is clear that either $C_2 = 0$ or $\sqrt{\frac{P}{EI}} = 0$. But for a given

pling load P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

$$\therefore \qquad C_2 = 0.$$

Now substituting the values of $C_1 = -\frac{M_0}{P}$ and $C_2 = 0$ in equation (*i*), we get

$$y = -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + 0 + \frac{M_0}{P}$$

$$= -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad ...(iii)$$

At the end B of the column, $x = l$ and $y = 0$.
Substituting these values in equation (*iii*), we get

$$0 = -\frac{M_0}{P} \cos\left(l \,.\, \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

$$\frac{M_0}{P} \cos\left(l \,.\, \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P}$$

$$\cos\left(l \,.\, \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P} \times \frac{P}{M_0} = 1 = \cos 0, \cos 2\pi, \cos 4\pi, \cos 6\pi, \ldots\ldots$$

$$\therefore \quad l \cdot \sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi, 6\pi, \ldots\ldots$$

Taking the least practical value,

$$l \cdot \sqrt{\frac{P}{EI}} = 2\pi \quad \text{or} \quad P = \frac{\pi^2 EI}{l^2} \qquad \ldots(19.3)$$

19.8. EXPRESSION FOR CRIPPLING LOAD WHEN ONE END OF THE COLUMN IS FIXED AND THE OTHER END IS HINGED (OR PINNED)

Consider a column AB of length l and uniform cross-sectional area fixed at the end A and hinged at the end B as shown in Fig. 19.7. Let P is the crippling load at which the column has buckled. Due to the crippling load P, the column will deflect as shown in Fig. 19.7.

There will be fixed end moment (M_0) at the fixed end A. This will try to bring back the slope of deflected column zero at A. Hence it will be acting anticlock wise at A. The fixed end moment M_0 at A is to be balanced. This will be balanced by a horizontal reaction (H) at the top end B as shown in Fig. 19.7.

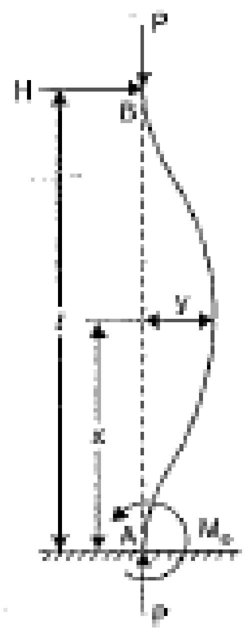

Fig. 19.7

Consider a section at a distance x from the end A.

Let y = Deflection of the column at the section,

M_0 = Fixed end moment at A, and

H = Horizontal reaction at B.

The moment at the section = Moment due to crippling load at B + Moment due to horizontal reaction at B

$$= -P \cdot y + H \cdot (l - x)$$

But the moment at the section is also

$$= EI \frac{d^2y}{dx^2}$$

Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = -P \cdot y + H(l - x)$$

or
$$EI \frac{d^2y}{dx^2} + P \cdot y = H(l - x)$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{H}{EI}(l - x) \qquad \text{(Dividing by } EI\text{)} \quad \ldots(A)$$

$$= \frac{H}{EI}(l - x) \times \frac{P}{P} = \frac{P}{EI} \cdot \frac{H(l - x)}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x) \quad ...(i)$$

where C_1 and C_2 are constants of integration and their values are obtained from boundary conditions. Boundary conditions are :

(i) At the fixed end A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$

(ii) At the hinged end B, $x = l$ and $y = 0$.

Substituting the value $x = 0$ and $y = 0$ in equation (i), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{H}{P}(l - 0) = C_1 + \frac{H \cdot l}{P}$$

$$\therefore \quad C_1 = -\frac{H}{P} \cdot l \quad ...(ii)$$

Differentiating the equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = C_1(-1)\sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

$$= -C_1 \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

At A, $x = 0$ and $\frac{dy}{dx} = 0$.

$$\therefore \quad 0 = -C_1 \times 0 + C_2 \cdot 1 \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P} \quad (\because \sin 0 = 0, \cos 0 = 1)$$

$$= C_2\sqrt{\frac{P}{EI}} - \frac{H}{P} \quad \text{or} \quad C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}.$$

Substituting the values of $C_1 = -\frac{H}{P} \cdot l$ and $C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}$ in equation (i), we get

$$y = -\frac{H}{P} \cdot l \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x)$$

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*The equation (A) can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 \cdot y = \frac{H}{EI}(l - x) \text{ where } \alpha^2 = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of this equation is

$$y = C_1 \cos(\alpha \cdot x) + C_2 \sin(\alpha \cdot x) + \frac{H(l - x)}{EI \times \alpha^2}$$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + \frac{H}{EI} \times \frac{(l - x)}{\left(\frac{P}{EI}\right)}$$

$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x)$$

At the end B, $x = l$ and $y = 0$.

Hence the above equation becomes as

$$0 = -\frac{H}{P}\, l \cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - l)$$

$$= -\frac{H}{P}\, l \cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) + 0$$

or $$\frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P}\, l \cos\left(l\sqrt{\frac{P}{EI}}\right)$$

or $$\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P} \cdot l \times \frac{P}{H} \times \sqrt{\frac{P}{EI}} \cdot \cos\left(l\sqrt{\frac{P}{EI}}\right)$$

$$= l \cdot \sqrt{\frac{P}{EI}} \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right)$$

or $$\tan\left(l\sqrt{\frac{P}{EI}}\right) = l \cdot \sqrt{\frac{P}{EI}}\,.$$

The solution to the above equation is, $l \cdot \sqrt{\frac{P}{EI}} = 4.5$ radians

Squaring both sides, we get

$$l^2 \cdot \frac{P}{EI} = 4.5^2 = 20.25$$

$$\therefore \quad P = 20.25\,\frac{EI}{l^2}$$

But approximately $20.25 = 2\pi^2$

$$\therefore \quad P = \frac{2\pi^2 EI}{l^2} \qquad ...(19.4)$$

19.9. EFFECTIVE LENGTH (OR EQUIVALENT LENGTH) OF A COLUMN

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

Let L_e = Effective length of a column,

l = Actual length of the column, and

P = Crippling load for the column.

Then the crippling load for any type of end condition is given by

$$P = \frac{\pi^2 EI}{L_e^{\,2}}. \qquad ...(19.5)$$

The crippling load (P) in terms of actual length and effective length and also the relation between effective length and actual length are given in Table 19.1.

TABLE 19.1

S.No.	*End conditions of column*	*Crippling load in terms of*		*Relation between effective length and actual length*
		Actual length	*Effective length*	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{{L_e}^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{{L_e}^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{{L_e}^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{{L_e}^2}$	$L_e = \frac{l}{\sqrt{2}}$

There are two values of moment of inertia *i.e.*, I_{xx} and I_{yy}.

The value of I (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

19.9.1. Crippling Stress in Terms of Effective Length and Radius of Gyration. The moment of inertia (I) can be expressed in terms of radius of gyration (k) as

$$I = Ak^2 \text{ where } A = \text{Area of cross-section.}$$

As I is the least value of moment of inertia, then

k = Least radius of gyration of the column section.

Now crippling load P in terms of effective length is given by

$$P = \frac{\pi^2 EI}{{L_e}^2} = \frac{\pi^2 E \times Ak^2}{{L_e}^2} \qquad (\because \; I = Ak^2)$$

$$= \frac{\pi^2 E \times A}{\frac{{L_e}^2}{k^2}} = \frac{\pi^2 E \times A}{\left(\frac{L_e}{k}\right)^2} \qquad ...(19.6)$$

And the stress corresponding to crippling load is given by

$$\text{Crippling stress} = \frac{\text{Crippling load}}{\text{Area}} = \frac{P}{A}$$

$$= \frac{\pi^2 E \times A}{A\left(\frac{L_e}{k}\right)^2} \qquad \text{(Substituting the value of } P\text{)}$$

$$= \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2} \qquad ...(19.7)$$

19.9.2. Slenderness Ratio. The ratio of the actual length of a column to the least radius of gyration of the column, is known as slenderness ratio.

Mathematically, slenderness ratio is given by

$$\text{Slenderness ratio} = \frac{\text{Actual length}}{\text{Least radius of gyration}} = \frac{l}{k} \quad ...(19.8)$$

19.10. LIMITATION OF EULER'S FORMULA

From equation (19.6), we have

$$\text{Crippling stress} = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}$$

For a column with both ends hinged, $L_e = l$. Hence Crippling stress becomes as $= \dfrac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$,

where $\dfrac{l}{k}$ is slenderness ratio.

If the slenderness ratio $\left(i.e., \dfrac{l}{k}\right)$ is small, the crippling stress (or the stress at failure) will be high. But for the column material, the crippling stress cannot be greater than the crushing stress. Hence when the slenderness ratio is less than a certain limit, Euler's formula gives a value of crippling stress greater than the crushing stress. In the limiting case, we can find the value of l/k for which crippling stress is equal to crushing stress.

For example, for a mid steel column with both ends hinged,

Crushing stress = 330 N/mm^2

Young's modulus, $E = 2.1 \times 10^5$ N/mm^2.

Equating the crippling stress to the crushing stress corresponding to the minimum value of slenderness ratio, we get

Crippling stress = Crushing stress

or
$$\frac{\pi^2 E}{\left(\frac{l}{k}\right)^2} = 330 \quad \text{or} \quad \frac{\pi^2 \times 2.1 \times 10^5}{\left(\frac{l}{k}\right)^2} = 330$$

$$\therefore \quad \left(\frac{l}{k}\right)^2 = \frac{\pi^2 \times 2.1 \times 10^5}{330} = 6282$$

$$\therefore \quad \frac{l}{k} = \sqrt{6282} = 79.27, \text{ say } 80.$$

Hence, if the slenderness ratio is less than 80 for mild steel column with both ends hinged, then Euler's formula will not be valid.

Problem 19.1. *A solid round bar 3 m long and 5 cm in diameter is used as a strut with both ends hinged. Determine the crippling (or collapsing) load. Take $E = 2.0 \times 10^5$ N/mm^2.*

Sol. Given :

Length of bar, $l = 3 \text{ m} = 3000 \text{ mm}$

Diameter of bar, $d = 5 \text{ cm} = 50 \text{ mm}$

Young's modulus, $E = 2.0 \times 10^5 \text{ N/mm}^2$

Moment of inertia, $I = \frac{\pi}{64} \times 5^4 = 30.68 \text{ cm}^4 = 30.68 \times 10^4 \text{ mm}^4$

Let P = Crippling load.

As both the ends of the bar are hinged, hence the crippling load is given by equation (19.1).

$$\therefore \quad P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$$

$$= 67288 \text{ N} = \textbf{67.288 kN.} \quad \textbf{Ans.}$$

Problem 19.2. *For the problem 19.1 determine the crippling load, when the given strut is used with the following conditions :*

(i) One end of the strut is fixed and the other end is free

(ii) Both the ends of strut are fixed

(iii) One end is fixed and other is hinged.

Sol. Given :

The data from Problem 19.1, is $l = 3000$ mm, diameter = 50 mm, $E = 2.0 \times 10^5 \text{ N/mm}^2$ and $I = 30.68 \times 10^4 \text{ mm}^4$.

Let P = Crippling load.

(i) Crippling load when one end is fixed and other is free

Using equation (19.2), $P = \frac{\pi^2 EI}{4l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{4 \times 3000^2} = \textbf{16822 N.}$ **Ans.**

Alternate Method

The crippling load for any type of end condition is given by equation (19.5) as,

$$P = \frac{\pi^2 EI}{L_e^2} \quad \text{...(i)}$$

where L_e = Effective length.

The effective length (L_e) when one end is fixed and other end is free from Table 19.1 on page 819 is given as

$$L_e = 2l = 2 \times 3000 = 6000 \text{ mm}$$

Substituting the value of L in equation (*i*), we get

$$P = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{6000^2} = \textbf{16822 N.} \quad \textbf{Ans.}$$

(ii) Crippling load when both the ends are fixed

Using equation (19.3), $P = \frac{4\pi^2 EI}{l^2} = \frac{4\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$

$$= 269152 \text{ N} = \textbf{269.152 kN.} \quad \textbf{Ans.}$$

Problem 19.4 (a). *A simply supported beam of length 4 metre is subjected to a uniformly distributed load of 30 kN/m over the whole span and deflects 15 mm at the centre. Determine the crippling loads when this beam is used as a column with the following conditions :*

(i) one end fixed and other end hinged

(ii) both the ends pin jointed. (Annamalai University, 1990)

Sol. Given :

Length, $L = 4 \text{ m} = 4000 \text{ mm}$

Uniformly distributed load, $w = 30 \text{ kN/m} = 30,000 \text{ N/m}$

$$= \frac{30,000}{1000} \text{ N/mm} = 30 \text{ N/mm}$$

Deflection at the centre, $\delta = 15$ mm.

For a simply supported beam, carrying U.D.L. over the whole span, the deflection at the centre is given by,

$$\delta = \frac{5}{384} \times \frac{w \times L^4}{EI}$$

or

$$15 = \frac{5}{384} \times \frac{30 \times 4000^4}{EI}$$

$\therefore$

$$EI = \frac{5}{384} \times \frac{30 \times 4000^4}{15}$$

$$= \frac{5}{384} \times \frac{3 \times 256}{15} \times 10^{13} = \frac{2}{3} \times 10^{13} \text{ N mm}^2.$$

(i) Crippling load when the beam is used as a column with one end fixed and other end hinged.

The crippling load P for this case in terms of actual length is given by equation (19.4) as

$$P = \frac{2\pi^2 \times EI}{L_e^2}, \text{ where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{2 \times \pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \mathbf{8224.5\ kN.} \quad \textbf{Ans.}$$

(ii) Crippling load when both the ends are pin-jointed

This is given by equation (19.1) in terms of actual length as

$$P = \frac{2\pi^2 \times EI}{l^2} \quad \text{where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{\pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \mathbf{4112.25\ kN.} \quad \textbf{Ans.}$$

Problem 19.5. *A solid round bar 4 m long and 5 cm in diameter was found to extend 4.6 mm under a tensile load of 50 kN. This bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.0.*

Sol. Given :

Actual length of bar, $L = 4$ m $= 4000$ mm

Dia. of bar, $d = 5$ cm

$\therefore$ Area of bar, $A = \frac{\pi}{4} \times 5^2 = 6.25\pi \text{ cm}^2 = 6.25\pi \times 10^2 \text{ mm}^2 = 625\pi \text{ mm}^2$

Extension of bar, $\delta L = 4.6$ mm

Tensile load, $W = 50$ kN $= 50000$ N.

In this problem, the values of Young's modulus (E) is not given. But it can be calculated from the given data.

$$\therefore \text{Young's modulus, } E = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{\left(\frac{\text{Tensile load}}{\text{Area}}\right)}{\left(\frac{\text{Extension of bar}}{\text{Length of bar}}\right)}$$

$$\left(\because \text{ Stress} = \frac{\text{Load}}{\text{Area}} \text{ and strain} = \frac{\delta L}{L}\right)$$

$$= \frac{\left(\frac{W}{A}\right)}{\frac{\delta L}{L}} = \frac{W}{A} \times \frac{L}{\delta L} = \frac{50000}{625\,\pi} \times \frac{4000}{4.6} = 2.214 \times 10^4 \text{ N/mm}^2.$$

Since the strut is hinged at its both ends,

$\therefore$ Effective length, L_e = Actual length = 4000 mm

Let P = Crippling or buckling load.

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2}$$

$$= \frac{\pi^2 \times 2.214 \times 10^4 \times \frac{\pi}{64} \times 5^4 \times 10^4}{4000 \times 4000} \qquad \left(\because I = \frac{\pi}{64} \times 5^4 \times 10^4 \text{ mm}^4\right)$$

$= \mathbf{4189.99}$ **say 4190 N. Ans.**

And safe load $= \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{4190}{4} =$ **1047.5 N. Ans.**

Problem 19.12 (a). *Using Euler's formula, calculate the critical stresses for a series of struts having slenderness ratio of 40, 80, 120, 160 and 200 under the following conditions :*

(i) both ends hinged and

(ii) both ends fixed.

Take $E = 2.05 \times 10^5\ N/mm^2$. (Annamalai University, 1990)

Sol. Given :

(i) Critical stresses when both ends hinged

Slenderness ratios, $\frac{l}{k} = 40, 80, 120, 160$ and 200

The critical stress or crippling stress is given by equation (19.7).

$$\therefore \quad \text{Critical stress} = \frac{\pi^2 \times E}{\left(\frac{L_e}{k}\right)^2}$$

where L_e = Effective length

But for both ends hinged, $L_e = l$

where l = actual length.

$$\therefore \quad \text{Critical stress} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2} \qquad (\because \quad L = l) \ ...(ii)$$

When $\frac{l}{k} = 40$, the critical stress becomes as

$$= \frac{\pi^2 \times E}{40^2} = \frac{\pi^2 \times 2.05 \times 10^5}{1600} = \mathbf{1264.54\ N/mm^2}. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 80$, the critical stress becomes as

$$= \frac{\pi^2 \times E}{80^2} = \frac{\pi^2 \times 2.05 \times 10^5}{6400} = \mathbf{316.135\ N/mm^2}. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 120$, the critical stress becomes as

$$= \frac{\pi^2 \times E}{120^2} = \frac{\pi^2 \times 2.05 \times 10^5}{14400} = \mathbf{140.5\ N/mm^2}. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 160$, the critical stress

$$= \frac{\pi^2 \times E}{160^2} = \frac{\pi^2 \times 2.05 \times 10^5}{25600} = \mathbf{79.03\ N/mm^2}. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 200$, the critical stress

$$= \frac{\pi^2 \times E}{200^2} = \frac{\pi^2 \times 2.05 \times 10^5}{400{,}00} = \mathbf{50.58\ N/mm^2}. \quad \textbf{Ans.}$$

19.11. RANKINE'S FORMULA

In Art. 19.10, we have learnt that Euler's formula gives correct results only for very long columns. But what happens when the column is a short or the column is not a very long. On the basis of results of experiments performed by Rankine, he established an empirical formula which is applicable to all columns whether they are short or long. The empirical formula given by Rankine is known as Rankine's formula, which is given as

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} \quad \ldots(i)$$

where P = Crippling load by Rankine's formula

P_C = Crushing load = $\sigma_c \times A$

σ_c = Ultimate crushing stress

A = Area of cross-section

P_E = Crippling load by Euler's formula

$= \frac{\pi^2 EI}{L_e^2}$, in which L_e = Effective length

For a given column material the crushing stress σ_c is a constant. Hence the crushing load P_C (which is equal to $\sigma_c \times A$) will also be constant for a given cross-sectional area of the column. In equation (*i*), P_C is constant and hence value of P depends upon the value of P_E. But for a given column material and given cross-sectional area, the value of P_E depends upon the effective length of the column.

(*i*) *If the column is a short*, which means the value of L_e is small, then the value of P_E will be large. Hence the value of $\frac{1}{P_E}$ will be small enough and is negligible as compared to the value of $\frac{1}{P_C}$. Neglecting the value of $\frac{1}{P_E}$ in equation (*i*), we get

$$\frac{1}{P} \to \frac{1}{P_C} \quad \text{or} \quad P \to P_C.$$

Hence the crippling load by Rankine's formula for a short column is approximately equal to crushing load. In Art. 19.2.1 also we have seen that short columns fail due to crushing.

(*ii*) *If the column is long*, which means the value of L_e is large. Then the value of P_E will be small and the value of $\frac{1}{P_E}$ will be large enough compared with $\frac{1}{P_C}$. Hence the value of $\frac{1}{P_C}$ may be neglected in equation (*i*).

$$\therefore \quad \frac{1}{P} = \frac{1}{P_E} \quad \text{or} \quad P \to P_E.$$

Hence the crippling load by Rankine's formula for long columns is approximately equal to crippling load given by Euler's formula.

Hence the Rankine's formula $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E}$ gives satisfactory results for all lengths of columns, ranging from short to long columns.

Now the Rankine's formula is $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} = \frac{P_E + P_C}{P_C \cdot P_E}$.

Taking reciprocal to both sides, we have

$$P = \frac{P_C \cdot P_E}{P_E + P_C} = \frac{P_C}{1 + \frac{P_C}{P_E}}$$

(Dividing the numerator and denominator by P_E)

$$= \frac{\sigma_c \times A}{1 + \frac{\sigma_c \, . \, A}{\left(\frac{\pi^2 EI}{L_e^2}\right)}} \qquad \left(\because \; P_c = \sigma_c \, . \, A \text{ and } P_E = \frac{\pi^2 EI}{L_e^2}\right)$$

But $I = Ak^2$, where k = least radius of gyration

∴ The above equation becomes as

$$P = \frac{\sigma_c \, . \, A}{1 + \frac{\sigma_c \, . \, A \, . \, L_e^2}{\pi^2 E \, . \, Ak^2}} = \frac{\sigma_c \, . \, A}{1 + \frac{\sigma_c}{\pi^2 E} \, . \left(\frac{L_e}{k}\right)^2}$$

$$= \frac{\sigma_c \, . \, A}{1 + \alpha \, . \left(\frac{L_e}{k}\right)^2} \qquad ...(19.9)$$

where $\alpha = \frac{\sigma_c}{\pi^2 E}$ and is known as Rankine's constant.

The equation (19.9) gives crippling load by Rankine's formula. As the Rankine formula is empirical formula, the value of 'α' is taken from the results of the experiments and is not calculated from the values of σ_c and E.

The values of σ_c and α for different columns material are given below in Table 19.2.

TABLE 19.2

S. No.	*Material*	σ_c *in N/mm²*	α
1.	Wrought Iron	250	$\frac{1}{9000}$
2.	Cast Iron	550	$\frac{1}{1600}$
3.	Mild Steel	320	$\frac{1}{7500}$
4.	Timber	50	$\frac{1}{750}$

Problem 19.13. *The external and internal diameter of a hollow cast iron column are 5 cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the values of σ_c = 550 N/mm² and $\alpha = \frac{1}{1600}$ in Rankine's formula.*

Sol. Given :

External dia., D = 5 cm

Internal dia., d = 4 cm

∴ Area, $A = \frac{\pi}{4}(5^2 - 4^2) = 2.25\pi \text{ cm}^2 = 2.25\pi \times 10^2 \text{ mm}^2 = 225\pi \text{ mm}^2$

Moment of Inertia, $I = \frac{\pi}{64}[5^4 - 4^4] = 5.7656\,\pi \text{ cm}^4$

$= 5.7656\pi \times 10^4 \text{ mm}^4 = 57656\pi \text{ mm}^4$

∴ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\pi}{225\pi}} = 25.625 \text{ mm}$$

Length of column, l = 3 m = 3000 mm

As both the ends are fixed,

∴ Effective length, $L_e = \frac{l}{2} = \frac{3000}{2} = 1500$ mm

Crushing stress, $\sigma_c = 550$ N/mm^2

Rankine's constant, $a = \frac{1}{1600}$

Let P = Crippling load by Rankine's formula

Using equation (19.9), we have

$$P = \frac{\sigma_c \cdot A}{1 + \left(\frac{L_e}{k}\right)^2} = \frac{550 \times 225\pi}{1 + \frac{1}{1600} \times \left(\frac{1500}{25.625}\right)^2}$$

$$= \frac{550 \times 225\pi}{3.1415} = \textbf{123750 N.} \quad \textbf{Ans.}$$

Problem 19.14. *A hollow cylindrical cast iron column is 4 m long with both ends fixed. Determine the minimum diameter of the column if it has to carry a safe load of 250 kN with a factor of safety of 5. Take the internal diameter as 0.8 times the external diameter. Take* $\sigma_c = 550$ *N/mm*^{2} *and* $a = \frac{1}{1600}$ *in Rankine's formula.* (AMIE, Winter 1983)

Sol. Given :

Length of column, l = 4 m = 4000 mm

End conditions = Both ends fixed

∴ Effective length, $L_e = \frac{l}{2} = \frac{4000}{2} = 2000$ mm

Safe load, = 250 kN

Factor of safety, = 5

Let External dia., = D

Internal dia. = $0.8 \times D$

Crushing stress, $\sigma_c = 550$ N/mm^2

Value of 'a' $= \frac{1}{1600}$ in Rankine's formula

Now factor of safety $= \frac{\text{Ciripling load}}{\text{Safe load}}$ or $5 = \frac{\text{Ciripling load}}{250}$

∴ Crippling load, $P = 5 \times 250 = 1250$ kN = 1250000 N

Area of column, $A = \frac{\pi}{4}[D^2 - (0.8D)^2]$

$$= \frac{\pi}{4}[D^2 - 0.64D^2] = \frac{\pi}{4} \times 0.36D^2 = \pi \times 0.09D^2$$

Moment of Inertia, $I = \frac{\pi}{64}[D^4 - (0.8D^4] = \frac{\pi}{64}[D^4 - 0.4096D^4]$

$$= \frac{\pi}{64} \times 0.5904 \times D^4 = 0.009225 \times \pi \times D^4$$

But $I = A \times k^2$, where k is radius of gyration

$$\therefore \quad k = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.009225 \times \pi D^4}{\pi \times 0.09 \times D^2}} = 0.32D$$

Now using equation (19.9), $P = \dfrac{\sigma_c \cdot A}{1 + a\left(\dfrac{L_e}{k}\right)^2}$

$$\text{or} \quad 1250000 = \frac{550 \times \pi \times 0.09\ D^2}{1 + \dfrac{1}{1600} \times \left(\dfrac{2000}{0.32D}\right)^2} \qquad (\because \ A = \pi \times 0.09D^2)$$

$$\frac{1250000}{550 \times \pi \times 0.09} = \frac{D^2}{1 + \dfrac{24414}{D^2}} \quad \text{or} \quad 8038 = \frac{D^2 \times D^2}{D^2 + 24414}$$

or $8038D^2 + 8038 \times 24414 = D^4$ or $D^4 - 8038D^2 - 8038 \times 24414 = 0$

or $D^4 - 8038\ D^2 - 196239700 = 0.$

The above equations is a quadratic equation in D^2. The solution is

$$\therefore \quad D^2 = \frac{8038 \pm \sqrt{8038^2 + 4 \times 1 \times 196239700}}{2}$$

$$\left(\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{8038 \pm \sqrt{646094 + 784958800}}{2} = \frac{8038 \pm 29147}{2}$$

$$= \frac{8038 + 29147}{2} \text{ (The other root is not possible)}$$

$$= 18592.5 \text{ mm}^2$$

$$\therefore \quad D = \sqrt{18592.5} = 136.3 \text{ mm}$$

$\therefore$ External diameter $=$ **136.3 mm. Ans.**

Internal diameter $= 0.8 \times 1363 =$ **109 mm. Ans.**

Problem 19.18. *A hollow cast iron column 200 mm outside diameter and 150 mm inside diameter, 8 m long has both ends fixed. It is subjected to an axial compressive load. Taking a factor of safety as 6, $\sigma_c = 560\ N/mm^2$, $a = \frac{1}{1600}$, determine the safe Rankine load.*

(AMIE, Summer 1990)

Sol. Given :

External dia., $D = 200$ mm

Internal dia., $d = 150$ mm

Length, $l = 8$ m $= 8000$ mm

End conditions = Both the ends are fixed

Crushing stress, $\sigma_c = 560\ N/mm^2$

Rankine's constant, $a = \frac{1}{1600}$

Safety factor = 6

Area of cross-section, $A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(200^2 - 150^2)$

$$= \frac{\pi}{4}(40000 - 22500) = 13744\ mm^2$$

Moment of inertia, $I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(200^4 - 150^4)$

$$= \frac{\pi}{64}(1600000000 - 506250000) = 53689000\ mm^4$$

Least radius of gyration, $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{53689000}{13744}} = 62.5$ mm

Let P = Crippling load by Rankine formula.

Using equation (19.9), $P = \dfrac{\sigma_c \times A}{1 + a\left(\frac{L_e}{k}\right)^2}$

where L_e = Effective length $= \frac{l}{2} = \frac{8000}{2} = 4000$ mm

$$\therefore \quad P = \frac{560 \times 13744}{1 + \frac{1}{1600} \times \left(\frac{4000}{62.5}\right)^2}$$

$$= \frac{7696640}{1 + 2.56} = \frac{7696640}{3.56} = 2161977\ N = 2161.977\ kN$$

$\therefore$ Safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{2161.977}{6} =$ **360.3295 kN. Ans.**