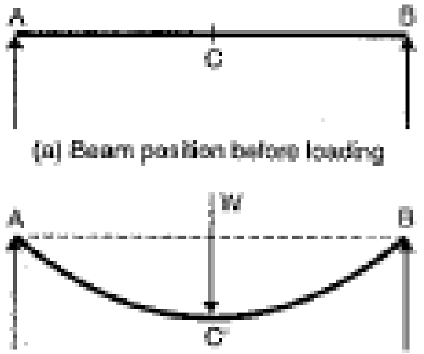
Deflection of Beams

12.1. INTRODUCTION

If a beam carries uniformly distributed load or a point load, the beam is deflected from its original position. In this chapter, we shall study the amount by which a beam is deflected from its position. Due to the loads acting on the beam, it will be subjected to bending moment. The radius of curvature of the deflected beam is given by the equation $\frac{M}{I} = \frac{E}{R}$. The radius of curvature will be constant if $R = \frac{I \times E}{M}$ = constant. The term $(I \times E)/M$ will be constant, if the beam is subjected to a constant bending moment M. This means that a beam for which, when loaded, the value of $(E \times I)/M$ is constant, will bend in a circular arc.

Fig. 12.1 (a) shows the beam position before any load is applied on the beam whereas Fig. 12.1 (b) shows the beam position after loading.



(a) Beam position before loading

(b) Beam position after loading

Fig. 12.1

12.2. DEFLECTION AND SLOPE OF A BEAM SUBJECTED TO UNIFORM BENDING MOMENT

A beam AB of length L is subjected to a uniform bending moment M as shown in Fig. 12.1 (c). As the beam is subjected to a constant bending moment, hence it will bend into a circular arc. The initial position of the beam is shown by ACB, whereas the deflected position is shown by $AC'B$.

Let R = Radius of curvature of the deflected beam,

y = Deflection of the beam at the centre (*i.e.*, distance CC'),

I = Moment of inertia of the beam section,

E = Young's modulus for the beam material, and

θ = Slope of the beam at the end A (*i.e.*, the angle made by the tangent at A with the beam AB). For a practical beam the deflection y is a small quantity.

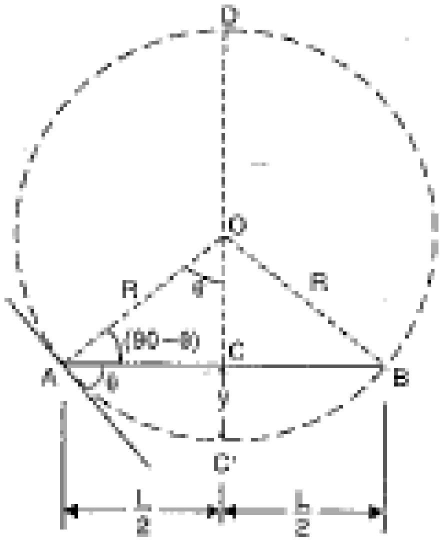


Fig. 12.1 (c)

Hence $\tan\theta = \theta$ where θ is in radians. Hence θ becomes the slope as slope is

$$\frac{dy}{dx} = \tan\theta = \theta.$$

Now $$AC = BC = \frac{L}{2}$$

Also from the geometry of a circle, we know that

$$AC \times CB = DC \times CC'$$

$$\frac{L}{2} \times \frac{L}{2} = (2R - y) \times y \qquad (\because \; DC = DC' - CC' = 2R - y)$$

or $$\frac{L^2}{4} = 2Ry - y^2$$

For a practical beam, the deflection y is a small quantity. Hence the square of a small quantity will be negligible. Hence neglecting y^2 in the above equation, we get

$$\frac{L^2}{4} = 2Ry$$

$\therefore$ $$y = \frac{L^2}{8R} \qquad ...(i)$$

But from bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

or $$R = \frac{E \times I}{M} \qquad ...(ii)$$

Substituting the value of R in equation (i), we get

$$y = \frac{L^2}{8 \times \frac{EI}{M}}$$

or $$y = \frac{ML^2}{8EI} \qquad ...(12.1)$$

The equation (12.1) gives the central deflection of a beam which bends in a circular arc.

Value of Slope (θ)

From triangle AOB, we know that

$$\sin\theta = \frac{AC}{AO} = \frac{\left(\frac{L}{2}\right)}{R} = \frac{L}{2R}$$

Since the angle θ is very small, hence $\sin\theta = \theta$ (in radians)

$\therefore$ $$\theta = \frac{L}{2R}$$

$$= \frac{L}{2 \times \frac{EI}{M}} \qquad \left(\because \; R = \frac{EI}{M} \text{ from equation } (ii)\right)$$

$$= \frac{M \times L}{2EI} \qquad ...(12.2)$$

Equation (12.2) gives the slope of the deflected beam at A or at B.

12.3. RELATION BETWEEN SLOPE, DEFLECTION AND RADIUS OF CURVATURE

Let the curve AB represents the deflection of a beam as shown in Fig. 12.2 (a). Consider a small portion PQ of this beam. Let the tangents at P and Q make angle ψ and $\psi + d\psi$ with x-axis. Normal at P and Q will meet at C such that

$$PC = QC = R$$

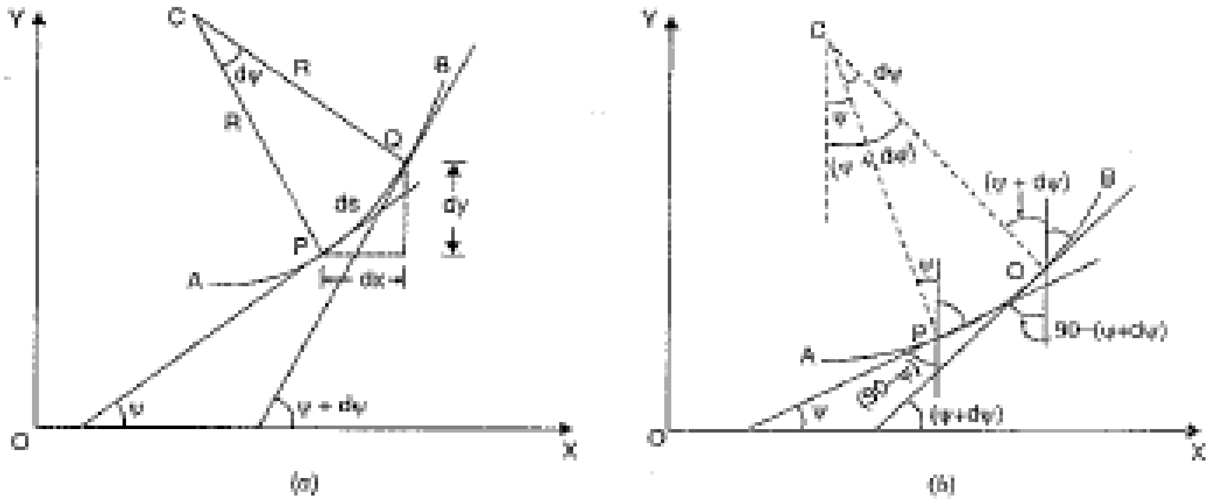


Fig. 12.2

The point C is known as centre of curvature of the curve PQ.

Let the length of PQ is equal to ds.

From Fig. 12.2 (b), we see that

$$\text{Angle } PCQ = d\psi$$

$$\therefore \quad PQ = ds = R.d\psi$$

or $$R = \frac{ds}{d\psi} \quad \ldots(i)$$

But if x and y be the co-ordinates of P, then

$$\tan\psi = \frac{dy}{dx} \quad \ldots(ii)$$

$$\sin\psi = \frac{dy}{ds}$$

and $$\cos\psi = \frac{dx}{ds}$$

Now equation (i) can be written as

$$R = \frac{ds}{d\psi} = \frac{\left(\frac{ds}{dx}\right)}{\left(\frac{d\psi}{dx}\right)} = \frac{\left(\frac{1}{\cos\psi}\right)}{\left(\frac{d\psi}{dx}\right)}$$

or $$R = \frac{\sec\psi}{\left(\frac{d\psi}{dx}\right)} \quad \ldots(iii)$$

Differentiating equation (*ii*) w.r.t. x, we get

$$\sec^2 \psi \cdot \frac{d\psi}{dx} = \frac{d^2 y}{dx^2}$$

or
$$\frac{d\psi}{dx} = \frac{\left(\frac{d^2 y}{dx^2}\right)}{\sec^2 \psi}$$

Substituting this value of $\frac{d\psi}{dx}$ in equation (*iii*), we get

$$R = \frac{\sec \psi}{\left(\frac{\frac{d^2 y}{dx^2}}{\sec^2 \psi}\right)} = \frac{\sec \psi \cdot \sec^2 \psi}{\frac{d^2 y}{dx^2}} = \frac{\sec^3 \psi}{\left(\frac{d^2 y}{dx^2}\right)}$$

Taking the reciprocal to both sides, we get

$$\frac{1}{R} = \frac{\frac{d^2 y}{dx^2}}{\sec^3 \psi} = \frac{\frac{d^2 y}{dx^2}}{(\sec^2 \psi)^{3/2}}$$

$$= \frac{\frac{d^2 y}{dx^2}}{(1 + \tan^2 \psi)^{3/2}}$$

For a practical beam, the slope $\tan \psi$ at any point is a small quantity. Hence $\tan^2 \psi$ can be neglected.

$$\therefore \quad \frac{1}{R} = \frac{d^2 y}{dx^2} \qquad \ldots(iv)$$

From the bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

or
$$\frac{1}{R} = \frac{M}{EI} \qquad \ldots(v)$$

Equating equations (*iv*) and (*v*), we get

$$\frac{M}{EI} = \frac{d^2 y}{dx^2}$$

$$\therefore \quad M = EI \frac{d^2 y}{dx^2} \qquad \ldots(12.3)$$

Differentiating the above equation w.r.t. x, we get

$$\frac{dM}{dx} = EI \frac{d^3 y}{dx^3}$$

But $\frac{dM}{dx} = F$ shear force (See page 288)

$$\therefore \quad F = EI \frac{d^3 y}{dx^3} \qquad \ldots(12.4)$$

Differentiating equation (12.4) w.r.t. x, we get

$$\frac{dF}{dx} = EI\frac{d^4y}{dx^4}$$

But $\frac{dF}{dx} = w$ the rate of loading

$\therefore$ $$w = EI\frac{d^4y}{dx^4} \quad ...(12.5)$$

Hence, the relation between curvature, slope, deflection etc. at a section is given by :

Deflection $= y$

Slope $= \frac{dy}{dx}$

Bending moment $= EI\frac{d^2y}{dx^2}$

Shearing force $= EI\frac{d^3y}{dx^3}$

The rate of loading $= EI\frac{d^4y}{dx^4}$.

Units. In the above equations, E is taken in N/mm^2

I is taken in mm^4, y is taken in mm,

M is taken in Nm and x is taken in m.

12.3.1. Methods of Determining Slope and Deflection at a Section in a Loaded Beam. The followings are the important methods for finding the slope and deflection at a section in a loaded beam :

(*i*) Double integration method

(*ii*) Moment area method, and

(*iii*) Macaulay's method

Incase of double integration method, the equation used is

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

First integration of the above equation gives the value of $\frac{dy}{dx}$ or slope. The second integration gives the value of y or deflection.

The first two methods are used for a single load whereas the third method is sued for several loads.

12.4. DEFLECTION OF A SIMPLY SUPPORTED BEAM CARRYING A POINT LOAD AT THE CENTRE

A simply supported beam AB of length L and carrying a point load W at the centre is shown in Fig. 12.3.

As the load is symmetrically applied the reactions R_A and R_B will be equal. Also the maximum deflection will be at the centre.

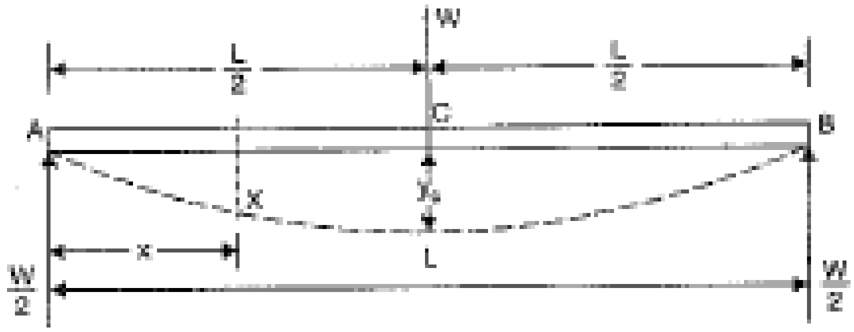


Fig. 12.3

Now $$R_A = R_B = \frac{W}{2}$$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x$$
$$= \frac{W}{2} \times x$$

(Plus sign is as B.M. for left portion at X is clockwise)

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = \frac{W}{2} \times x \quad \text{...(i)}$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1 \quad \text{...(ii)}$$

where C_1 is the constant of integration. And its value is obtained from boundary conditions. The boundary condition is that at $x = \frac{L}{2}$, slope $\left(\frac{dy}{dx}\right) = 0$ (As the maximum deflection is at the centre, hence slope at the centre will be zero). Substituting this boundary condition in equation (*ii*), we get

$$O = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + C_1$$

or $$C_1 = -\frac{WL^2}{16}$$

Substituting the value of C_1 in equation (*ii*), we get

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16} \quad \text{...(iii)}$$

The above equation is known the *slope equation*. We can find the slope at any point on the beam by substituting the values of x. Slope is maximum at A. At A, $x = 0$ and hence slope at A will be obtained by substituting $x = 0$ in equation (*iii*).

$$\therefore \quad EI\left(\frac{dy}{dx}\right)_{at\ A} = \frac{W}{4}\times 0 - \frac{WL^2}{16}$$

$\left[\left(\frac{dy}{dx}\right)_{at\ A}\right.$ is the slope at A and is represented by $\left.\theta_A\right]$

or
$$EI \times \theta_A = -\frac{WL^2}{16}$$

$$\therefore \quad \theta_A = -\frac{WL^2}{16EI}$$

The slope at point B will be equal to θ_A, since the load is symmetrically applied.

$$\therefore \quad \theta_B = \theta_A = -\frac{WL^2}{16EI} \qquad ...(12.6)$$

Equation (12.6) gives the slope in radians.

Deflection at any point

Deflection at any point is obtained by integrating the slope equation (*iii*). Hence integrating equation (*iii*), we get

$$EI \times y = \frac{W}{4}\cdot\frac{x^3}{3} - \frac{WL^2}{16}x + C_2 \qquad ...(iv)$$

where C_2 is another constant of integration. At A, $x = 0$ and the deflection (y) is zero.

Hence substituting these values in equation (*iv*), we get

$$EI \times 0 = 0 - 0 + C_2$$

or
$$C_2 = 0$$

Substituting the value of C_2 in equation (*iv*), we get

$$EI \times y = \frac{Wx^3}{12} - \frac{WL^2 \cdot x}{16} \qquad ...(v)$$

The above equation is known as *the deflection equation*. We can find the deflection at any point on the beam by substituting the values of x. The deflection is maximum at centre point C, where $x = \frac{L}{2}$. Let y_c represents the deflection at C. Then substituting $x = \frac{L}{2}$ and $y = y_c$ in equation (v), we get

$$EI \times y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16}\times\left(\frac{L}{2}\right)$$

$$= \frac{WL^3}{96} - \frac{WL^3}{32} = \frac{WL^3 - 3WL^3}{96}$$

$$= -\frac{2WL^3}{96} = -\frac{WL^3}{48}$$

$$\therefore \quad y_c = -\frac{WL^3}{48EI}$$

(Negative sign shows that deflection is downwards)

$$\therefore \quad \text{Downward deflection, } y_c = \frac{WL^3}{48EI} \qquad ...(12.7)$$

Problem 12.1. *A beam 6 m long, simply supported at its ends, is carrying a point load of 50 kN at its centre. The moment of inertia of the beam (i.e. I) is given as equal to 78 × 10⁶ mm⁴. If E for the material of the beam = 2.1 × 10⁵ N/mm², calculate : (i) deflection at the centre of the beam and (ii) slope at the supports.*

Sol. Given :

Length, $L = 6 \text{ m} = 6 \times 1000 = 6000 \text{ mm}$

Point load, $W = 50 \text{ kN} = 50,000 \text{ N}$

M.O.I., $I = 78 \times 10^6 \text{ mm}^4$

Value of $E = 2.1 \times 10^5 \text{ N/mm}^2$

Let y_c = Deflection at the centre and

θ_A = Slope at the support.

(*i*) Using equation (12.7) for the deflection at the centre, we get

$$y_c = \frac{WL^3}{48EI}$$

$$= \frac{50000 \times 6000^3}{48 \times 2.1 \times 10^5 \times 78 \times 10^6}$$

$$= \mathbf{13.736 \text{ mm.}} \quad \textbf{Ans.}$$

(*ii*) Using equation (12.6) for the slope at the supports, we get

$$\theta_B = \theta_A = -\frac{WL^2}{16EI}$$

$$= \frac{WL^2}{16EI} \qquad \text{(Numerically)}$$

$$= \frac{50000 \times 6000^2}{16 \times 2.1 \times 10^5 \times 78 \times 10^6} \text{ radians}$$

$$= 0.06868 \text{ radians}$$

$$= 0.06868 \times \frac{180}{\pi} \text{ degree} \qquad \left(\because \; 1 \text{ radian} = \frac{180}{\pi} \text{ degree}\right)$$

$$= \mathbf{3.935°.} \quad \textbf{Ans.}$$

Problem 12.2. *A beam 4 metre long, simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1°, find the deflection at the centre of the beam.*

Sol. Given :

Length, $L = 4 \text{ m} = 4000 \text{ mm}$

Point load at centre = W

Slope at the ends, $\theta_A = \theta_B = 1° = \frac{1 \times \pi}{180} = 0.01745$ radians

Let y_c = Deflection at the centre

Using equation (12.6), for the slope at the supports, we get

$$\theta_A = \frac{WL^2}{16EI} \qquad \text{(Numerically)}$$

or

$$0.01745 = \frac{WL^2}{16EI} \qquad \ldots(i)$$

Now using equation (12.7), we get

$$y_c = \frac{WL^3}{48EI}$$

$$= \frac{WL^2}{16EI} \times \frac{L}{3} \qquad \left(\because \frac{WL^3}{48EI} = \frac{WL^2}{16EI} \times \frac{L}{3}\right)$$

$$= 0.01745 \times \frac{4000}{3} \qquad \left[\because \frac{WL^2}{16EI} = 0.01745 \text{ from equation } (i)\right]$$

$$= \mathbf{23.26\ mm.} \quad \textbf{Ans.}$$

Problem 12.3. *A beam 3 m long, simply supported at its ends, is carrying a point load W at the centre. If the slope at the ends of the beam should not exceed 1°, find the deflection at the centre of the beam.* (Annamalai University, 1991)

Sol. Given :

Length, $L = 3 \text{ m} = 3 \times 1000 = 3000$ mm

Point load at centre $= W$

Slope at the ends, $\theta_A = \theta_B = 1°$

$$= \frac{1 \times \pi}{180} = 0.01745 \text{ radians}$$

Let y_c = Deflection at the centre

Using equation (12.6), we get

$$\theta_A = \frac{WL^2}{16EI} \quad \text{or} \quad 0.01745 = \frac{WL^2}{16EI} \qquad \ldots(i)$$

Now using equation (12.7), we get

$$y_c = \frac{WL^3}{48EI} = \frac{WL^2}{16EI} \times \frac{L}{3}$$

$$= 0.01745 \times \frac{L}{3} \qquad \left(\because \frac{WL^2}{16EI} = 0.01745\right)$$

$$= 0.01745 \times \frac{3000}{3} \qquad (\because L = 3000 \text{ mm})$$

$$= \mathbf{17.45\ mm.} \quad \textbf{Ans.}$$

$$y_C = \frac{5}{384} \cdot \frac{W \cdot L^3}{EI} \qquad ...(12.14)$$

Problem 12.5. *A beam of uniform rectangular section 200 mm wide and 300 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9 kN/m run over the entire span of 5 m. If the value of E for the beam material is 1 × 10⁴ N/mm², find :*

(i) the slope at the supports and *(ii) maximum deflection.*

Sol. Given :

Width, $b = 200$ mm

Depth, $d = 300$ mm

M.O.I., $I = \frac{bd^3}{12} = \frac{200 \times 300^3}{12} = 4.5 \times 10^8 \text{ mm}^4$

U.d.l., $w = 9$ kN/m $= 9000$ N/m

Span, $L = 5$ m $= 5000$ mm

∴ Total load, $W = w \cdot L^* = 9000 \times 5 = 45000$ N

Value of $E = 1 \times 10^4$ N/mm²

Let θ_A = Slope at the support

and y_C = Maximum deflection.

(*i*) Using equation (12.12), we get

$$\theta_A = -\frac{W \cdot L^2}{24EI}$$

$$= -\frac{45000 \times 5000^2}{24 \times 1 \times 10^4 \times 4.5 \times 10^8} \text{ radians}$$

= **0.0104 radians. Ans.**

(*ii*) Using equation (12.14), we get

$$y_C = \frac{5}{384} \cdot \frac{W \cdot L^3}{EI}$$

$$= \frac{5}{384} \times \frac{45000 \times 5000^3}{1 \times 10^4 \times 4.5 \times 10^8}$$

= **16.27 mm. Ans.**

Problem 12.7. *A beam of length 5 m and of uniform rectangular section is supported at its ends and carries uniformly distributed load over the entire length. Calculate the depth of the section if the maximum permissible bending stress is 8 N/mm² and central deflection is not to exceed 10 mm.*

Take the value of $E = 1.2 \times 10^4$ N/mm².

Sol. Given :

Length, $L = 5$ m $= 5000$ mm

Bending stress, $f = 8$ N/mm²

Central deflection, $y_C = 10$ mm

Value of $E = 1.2 \times 10^4$ N/mm²

Let W = Total load

and d = Depth of beam

The maximum bending moment for a simply supported beam carrying a uniformly distributed load is given by,

$$M = \frac{w.L^2}{8} = \frac{W.L}{8} \quad (\because W = w.L) \quad ...(i)$$

Now using the bending equation,

$$\frac{M}{I} = \frac{f}{y}$$

or $$M = \frac{f \times I}{y} = \frac{8 \times I}{(d/2)} \quad \left(\because y = \frac{d}{2}\right)$$

$\therefore$ $$M = \frac{16I}{d} \quad ...(ii)$$

Equating the two values of B.M., we get

$$\frac{W.L}{8} = \frac{16I}{d}$$

or $$W = \frac{16 \times 8I}{L \times d} = \frac{128I}{L \times d} \quad ...(iii)$$

Now using equation (12.14), we get

$$y_C = \frac{5}{384} \times \frac{WL^3}{EI}$$

or $$10 = \frac{5}{384} \times \frac{128I}{L \times d} \times \frac{L^3}{EI} \quad \left(\because y_C = 10 \text{ mm and } W = \frac{128I}{L \times d}\right)$$

$$= \frac{5}{384} \times \frac{128 \times L^2}{d \times E}$$

or $$d = \frac{5}{384} \times \frac{128 \times L^2}{10 \times E} = \frac{5}{384} \times \frac{128 \times 5000^2}{10 \times 1.2 \times 10^4}$$

$$= 347.2 \text{ mm} = 34.72 \text{ cm.} \quad \textbf{Ans.}$$

12.7. MACAULAY'S METHOD

The procedure of finding slope and deflection for a simply supported beam with an eccentric point load as mentioned in Art. 12.5, is a very laborious. There is a convenient method for determining the deflections of the beam subjected to point loads.

This method was devised by Mr. M.H. Macaulay and is known as Macaulay's method. This method mainly consists in the special manner in which the bending moment at any section is expressed and in the manner in which the integrations are carried out.

12.7.1. Deflection of a Simply Supported Beam with an Eccentric Point Load. A simply supported beam AB of length L and carrying a point load W at a distance 'a' from left support and at a distance 'b' from right support is shown in Fig. 12.7. The reactions at A and B are given by,

$$R_A = \frac{W.b}{L} \quad \text{and} \quad R_B = \frac{W.a}{L}$$

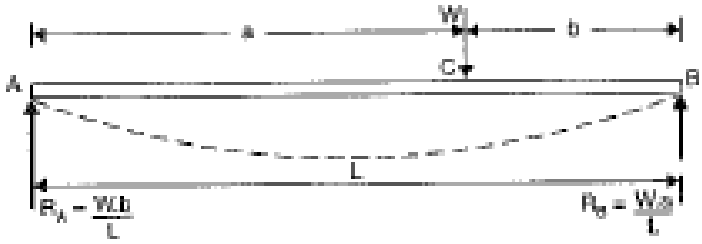


Fig. 12.7

The bending moment at any section between A and C at a distance x from A is given by,

$$M_x = R_A \times x = \frac{W.b}{L} \times x$$

The above equation of B.M. holds good for the values of x between 0 and 'a'. The B.M. at any section between C and B at a distance x from A is given by,

$$M_x = R_A x - W \times (x - a)$$

$$= \frac{W.b}{L} . x - W(x - a)$$

The above equation of B.M. holds good for all values of x between $x = a$ and $x = b$.

The B.M. for all sections of the beam can be expressed in a single equation written as

$$M_x = \frac{W.b}{L} x \;\vdots\; - W(x - a) \qquad ...(i)$$

Stop at the dotted line for any point in section AC. But for any point in section CB, add the expression beyond the dotted line also.

The B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2 y}{dx^2} \qquad ...(ii)$$

Hence equating (i) and (ii), we get

$$EI \frac{d^2 y}{dx^2} = \frac{W.b}{L} . x \;\vdots\; - W(x - a) \qquad ...(iii)$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{W.b}{L} \frac{x^2}{2} + C_1 \;\vdots\; - \frac{W(x - a)^2}{2} \qquad ...(iv)$$

where C_1 is a constant of integration. This constant of integration should be written after the first term. Also *the brackets are to be integrated as a whole*. Hence the integration of $(x - a)$ will be $\frac{(x-a)^2}{2}$ and not $\frac{x^2}{2} - ax$.

Integrating equation (*iv*) once again, we get

$$EIy = \frac{W.b}{2L} \cdot \frac{x^3}{3} + C_1x + C_2 \quad \vdots \quad - \frac{W}{2} \frac{(x-a)^3}{3} \quad \text{...(v)}$$

where C_2 is another constant of integration. This constant is written after C_1x. The integration of $(x - a)^2$ will be $\left(\frac{x-a}{3}\right)^3$. This type of integration is justified as the constant of integrations C_1 and C_2 are valid for all values of x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(*i*) At $x = 0, y = 0$ and (*ii*) At $x = L, y = 0$

(*i*) At A, $x = 0$ and $y = 0$. Substituting these values in equation (*v*) upto dotted line only, we get

$$0 = 0 + 0 + C_2$$

$$\therefore \quad C_2 = 0$$

(*ii*) At B, $x = L$ and $y = 0$. Substituting these values in equation (*v*), we get

$$0 = \frac{W.b}{2L} \cdot \frac{L^3}{3} + C_1 \times L + 0 - \frac{W}{2} \frac{(L-a)^3}{3}$$

($\because$ $C_2 = 0$. Here complete Eq. (*v*) is to be taken)

$$= \frac{W.b.L^2}{6} + C_1 \times L - \frac{W}{2} \frac{b^3}{3} \quad (\because \; L - a = b)$$

$$\therefore \quad C_1 \times L = \frac{W}{6} \cdot b^3 - \frac{W.b.L^2}{6} = -\frac{W.b}{6} (L^2 - b^2)$$

$$\therefore \quad C_1 = -\frac{W.b}{6L} (L^2 - b^2) \quad \text{...(vi)}$$

Substituting the value of C_1 in equation (*iv*), we get

$$EI \frac{dy}{dx} = \frac{W.b}{L} \frac{x^2}{2} + \left[-\frac{W.b}{6L}\left(L^2 - b^2\right)\right] \quad \vdots \quad - \frac{W(x-a)^2}{2}$$

$$= \frac{W.b.x^2}{2L} - \frac{W.b}{6L}\left(L^2 - b^2\right) \quad \vdots \quad - \frac{W(x-a)^2}{2} \quad \text{...(vii)}$$

The equation (*vii*) gives the slope at any point in the beam. Slope is maximum at A or B. To find the slope at A, substitute $x = 0$ in the above equation upto dotted line as point A lies in AC.

$$\therefore \quad EI.\theta_A = \frac{W.b}{2L} \times 0 - \frac{Wb}{6L} (L^2 - b^2) \quad \left(\because \frac{dy}{dx} \text{ at } A = \theta_A\right)$$

$$= -\frac{Wb}{6L} (L^2 - b^2)$$

$$\therefore \quad \theta_A = -\frac{Wb}{6EIL} (L^2 - b^2) \quad \text{(as given before)}$$

Substituting the values of C_1 and C_2 in equation (v), we get

$$EIy = \frac{W \cdot b}{6L} \cdot x^3 + \left[-\frac{Wb}{6L}(L^2 - b^2) \right] x + 0 \quad \vdots \quad -\frac{W}{6}(x-a)^3 \qquad ...(viii)$$

The equation ($viii$) gives the deflection at any point in the beam. To find the deflection y_c under the load, substitute $x = a$ in equation ($viii$) and consider the equation upto dotted line (as point C lies in AC). Hence, we get

$$EIy_c = \frac{W \cdot b}{6L} \cdot a^3 - \frac{W \cdot b}{6L}(L^2 - b^2)a = \frac{W \cdot b}{6L} \cdot a\,(a^2 - L^2 + b^2)$$

$$= -\frac{W \cdot a \cdot b}{6L}(L^2 - a^2 - b^2)$$

$$= -\frac{W \cdot a \cdot b}{6L}[(a+b)^2 - a^2 - b^2] \qquad (\because \; L = a + b)$$

$$= -\frac{W \cdot a \cdot b}{6L}[a^2 + b^2 + 2ab - a^2 - b^2]$$

$$= -\frac{W \cdot a \cdot b}{6L}[2ab] = -\frac{Wa^2 \cdot b^2}{3L}$$

$$\therefore \qquad y_c = -\frac{Wa^2 \cdot b^2}{3EIL} \qquad ...\text{(same as before)}$$

Note. While using Macaulay's Method, the section x is to be taken in the last portion of the beam.

Problem 12.8. *A beam of length 6 m is simply supported at its ends and carries a point load of 40 kN at a distance of 4 m from the left support. Find the deflection under the load and maximum deflection. Also calculate the point at which maximum deflection takes place. Given M.O.I. of beam = 7.33 × 10⁷ mm⁴ and E = 2 × 10⁵ N/mm².*

Sol. Given :

Length, $L = 6$ m $= 6000$ mm

Point load, $W = 40$ kN $= 40{,}000$ N

Distance of point load from left support, $a = 4$ m $= 4000$ mm

$\therefore$ $b = L - a = 6 - 4 = 2$ m $= 2000$ mm

Let y_c = Deflection under the load

y_{max} = Maximum deflection

Using equation $y_c = -\dfrac{W \cdot a^2 \cdot b^2}{3EIL}$

$$\therefore \qquad y_c = -\frac{40000 \times 4000^2 \times 2000^2}{3 \times 2 \times 10^5 \times 7.33 \times 10^7 \times 6000}$$

$$= -\mathbf{9.7\ mm.} \quad \textbf{Ans.}$$

Problem 12.9. *A beam of length 6 m is simply supported at its ends and carries two point loads of 48 kN and 40 kN at a distance of 1 m and 3 m respectively from the left support. Find :*

(i) deflection under each load,

(ii) maximum deflection, and

(iii) the point at which maximum deflection occurs.

Given $E = 2 \times 10^5$ N/mm² and $I = 85 \times 10^6$ mm⁴.

Sol. Given :

$$I = 85 \times 10^6 \text{ mm}^4 \; ; \; E = 2 \times 10^5 \text{ N/mm}^2$$

First calculate the reactions R_A and R_B.

Taking moments about A, we get

$$R_B \times 6 = 48 \times 1 + 40 \times 3 = 168$$

$$\therefore \quad R_B = \frac{168}{6} = 28 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load} - R_B = (48 + 40) - 28 = 60 \text{ kN}$$

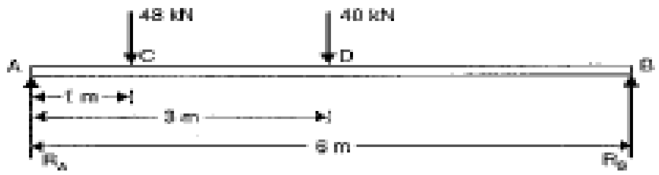


Fig. 12.8

Consider the section X in the last part of the beam (*i.e.*, in length DB) at a distance x from the left support A. The B.M. at this section is given by,

$$EI \frac{d^2y}{dx^2} = R_A x \;\vdots\; - 48(x-1) \;\vdots\; - 40(x-3)$$

$$= 60x \;\vdots\; - 48(x-1) \;\vdots\; - 40(x-3)$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{60x^2}{2} + C_1 \;\vdots\; - 48 \frac{(x-1)^2}{2} \;\vdots\; - 40 \frac{(x-3)^2}{2}$$

$$= 30x^2 + C_1 \;\vdots\; - 24(x-1)^2 \;\vdots\; - 20(x-3)^2 \qquad \text{...(i)}$$

Integrating the above equation again, we get

$$EIy = \frac{30x^3}{3} + C_1 x + C_2 \;\vdots\; \frac{-24(x-1)^3}{3} \;\vdots\; \frac{-20(x-3)^3}{3}$$

$$= 10x^3 + C_1 x + C_2 \;\vdots\; - 8(x-1)^3 \;\vdots\; - \frac{20}{3}(x-3)^3 \qquad \text{...(ii)}$$

To find the values of C_1 and C_2, use two boundary conditions. The boundary conditions are :

(*i*) at $x = 0$, $y = 0$, and (*ii*) at $x = 6$ m, $y = 0$.

(*i*) Substituting the first boundary condition *i.e.*, at $x = 0$, $y = 0$ in equation (*ii*) and considering the equation upto first dotted line (as $x = 0$ lies in the first part of the beam), we get

$$0 = 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(*ii*) Substituting the second boundary condition *i.e.*, at $x = 6$ m, $y = 0$ in equation (*ii*) and considering the complete equation (as $x = 6$ lies in the last part of the beam), we get

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3 \qquad (\because \; C_2 = 0)$$

or

$$0 = 2160 + 6C_1 - 8 \times 5^3 - \frac{20}{3} \times 3^3$$

$$= 2160 + 6C_1 - 1000 - 180 = 980 + 6C_1$$

$$\therefore \quad C_1 = \frac{-980}{6} = -163.33$$

Now substituting the values of C_1 and C_2 in equation (*ii*), we get

$$EIy = 10x^3 - 163.33x \;\vdots\; - 8(x-1)^3 \;\vdots\; - \frac{20}{3}(x-3)^3 \qquad ...(iii)$$

(*i*) (*a*) *Deflection under first load i.e., at point C.* This is obtained by substituting $x = 1$ in equation (*iii*) upto the first dotted line (as the point C lies in the first part of the beam). Hence, we get

$$\begin{aligned} EI.\, y_c &= 10 \times 1^3 - 163.33 \times 1 \\ &= 10 - 163.33 = -153.33 \text{ kNm}^3 \\ &= -153.33 \times 10^3 \text{ Nm}^3 \\ &= -153.33 \times 10^3 \times 10^9 \text{ Nmm}^3 \\ &= -153.33 \times 10^{12} \text{ Nmm}^3 \end{aligned}$$

$$\therefore \quad y_c = \frac{-153.33 \times 10^{12}}{EI} = \frac{-153.33 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} \text{ mm}$$

$$= -\mathbf{9.019 \text{ mm.}} \quad \textbf{Ans.}$$

(Negative sign shows that deflection is downwards).

(*b*) *Deflection under second load i.e. at point D.* This is obtained by substituting $x = 3$ m in equation (*iii*) upto the second dotted line (as the point D lies in the second part of the beam). Hence, we get

$$\begin{aligned} EI\, y_D &= 10 \times 3^3 - 163.33 \times 3 - 8(3-1)^3 \\ &= 270 - 489.99 - 64 = -283.99 \text{ kNm}^3 \\ &= -283.99 \times 10^{12} \text{ Nmm}^3 \end{aligned}$$

$$\therefore \quad y_D = \frac{-283.99 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -\mathbf{16.7 \text{ mm.}} \quad \textbf{Ans.}$$

(*ii*) *Maximum Deflection.* The deflection is likely to be maximum at a section between C and D. For maximum deflection, $\frac{dy}{dx}$ should be zero. Hence equate the equation (*i*) equal to zero upto the second dotted line.

$$\therefore \quad 30x^2 + C_1 - 24(x-1)^2 = 0$$

or $\quad 30x^2 - 163.33 - 24(x^2 + 1 - 2x) = 0 \qquad (\because \; C_1 = -163.33)$

or $\quad 6x^2 + 48x - 187.33 = 0$

The above equation is a quadratic equation. Hence its solution is

$$x = \frac{-48 \pm \sqrt{48^2 + 4 \times 6 \times 187.33}}{2 \times 6} = 2.87 \text{ m.}$$

(Neglecting – ve root)

Now substituting $x = 2.87$ m in equation (*iii*) upto the second dotted line, we get maximum deflection as

$$\begin{aligned} EIy_{max} &= 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2.87 - 1)^3 \\ &= 236.39 - 468.75 - 52.31 \\ &= 284.67 \text{ kNm}^3 = -284.67 \times 10^{12} \text{ Nmm}^3 \end{aligned}$$

$$\therefore \quad y_{max} = \frac{-284.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -\mathbf{16.745 \text{ mm.}} \quad \textbf{Ans.}$$

12.8. MOMENT AREA METHOD

Fig. 12.17 shows a beam AB carrying some type of loading, and hence subjected to bending moment as shown in Fig. 12.17 (a). Let the beam bent into AQ_1P_1B as shown in Fig. 12.17 (b).

Due to the load acting on the beam. Let A be a point of zero slope and zero deflection.

Consider an element PQ of small length dx at a distance x from B. The corresponding points on the deflected beam are P_1Q_1 as shown in Fig. 12.17 (b).

Let R = Radius of curvature of deflected part P_1Q_1

$d\theta$ = Angle subtended by the arc P_1Q_1 at the centre O

M = Bending moment between P and Q

P_1C = Tangent at point P_1

Q_1D = Tangent at point Q_1.

The tangent at P_1 and Q_1 are cutting the vertical line through B at points C and D. The angle between the normals at P_1 and Q_1 will be equal to the angle

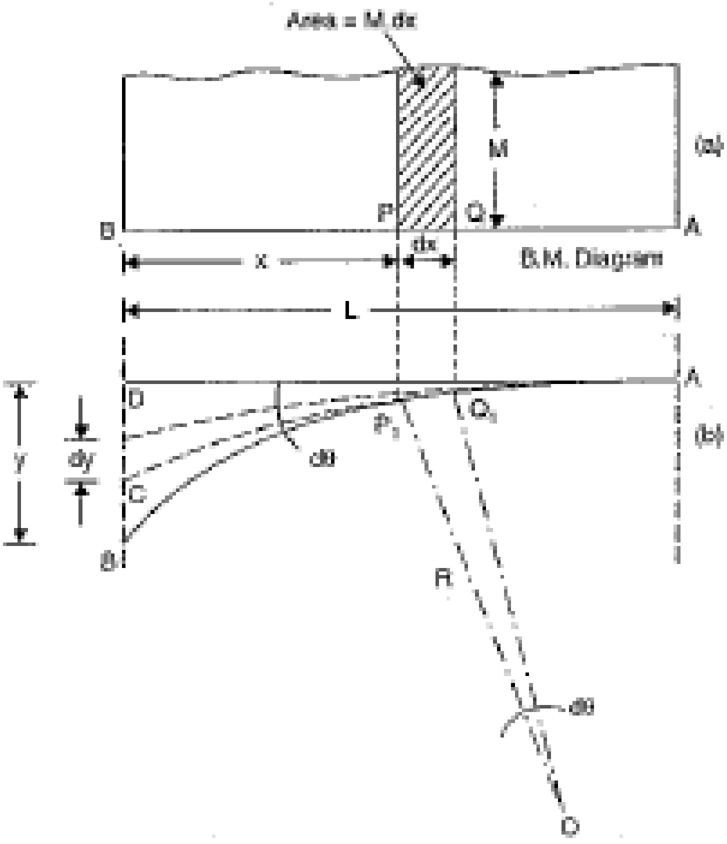


Fig. 12.17

between the tangents at P_1 and Q_1. Hence the angle between the lines CP_1 and DQ_1 will be equal to $d\theta$.

For the deflected part P_1Q_1 of the beam, we have

$$P_1Q_1 = R.d\theta$$

But $$P_1Q_1 = dx$$

$\therefore$ $$dx = R.d\theta$$

$\therefore$ $$d\theta = \frac{dx}{R} \quad ...(i)$$

But for a loaded beam, we have

$$\frac{M}{I} = \frac{E}{R} \quad \text{or} \quad R = \frac{EI}{M}$$

Substituting the values of R in equation (i), we get

$$d\theta = \frac{dx}{\left(\frac{EI}{M}\right)} = \frac{M\,dx}{EI} \quad ...(ii)$$

Since the slope at point A is assumed zero, hence total slope at B is obtained by integrating the above equation between the limits 0 and L.

$\therefore$ $$\theta = \int_0^L \frac{M.dx}{EI} = \frac{1}{EI}\int_0^L M.dx$$

But $M.dx$ represents the area of B. M. diagram of length dx. Hence $\int_0^L M.dx$ represents the area of B. M. diagram between A and B.

$\therefore$ $$\theta = \frac{1}{EI}\,[\text{Area of B. M. diagram between } A \text{ and } B]$$

But $$\theta = \text{slope at } B = \theta_B$$

$\therefore$ Slope at B,

$$\theta_B = \frac{\text{Area of B. M. diagram between } A \text{ and } B}{EI} \quad ...(12.15)$$

If the slope at A is not zero then, we have

"Total change of slope between B and A is equal to the area of B. M. diagram between B and A divided by the flexural rigidity EI"

or $$\theta_B - \theta_A = \frac{\text{Area of B.M. between } A \text{ and } B}{EI} \quad ...(12.16)$$

Now the deflection, due to bending of the portion P_1Q_1 is given by

$$dy = x.d\theta$$

Substituting the value of $d\theta$ from equation (ii), we get

$$dy = x\,.\,\frac{M.dx}{EI} \quad ...(iii)$$

Since deflection at A is assumed to be zero, hence the total deflection at B is obtained by integrating the above equation between the limits zero and L.

$\therefore$ $$y = \int_0^L \frac{xM.dx}{EI} = \frac{1}{EI}\int_0^L xM.dx$$

But $x \times M.dx$ represents the moment of area of the B.M. diagram of length dx about point B.

Hence $\int_0^L xM.dx$ represents the moment of area of the B.M. diagram between B and A about B. This is equal to the total area of B.M. diagram between B and A multiplied by the distance of the C.G. of the B.M. diagram area from B.

$\therefore$ $$y = \frac{1}{EI} \times A \times \bar{x} = \frac{A\bar{x}}{EI} \quad ...(12.17)$$

where A = Area of B.M. diagram between A and B

$\bar{x}$ = Distance of C.G. of the area A from B.

Deflection of Cantilevers

13.1. INTRODUCTION

Cantilever is a beam whose one end is fixed and other end is free. In this chapter we shall discuss the methods of finding slope and deflection for the cantilevers when they are subjected to various types of loading. The important methods are (*i*) Double integration method (*ii*) Macaulay's method and (*iii*) Moment-area-method. These methods have also been used for finding deflections and slope of the simply supported beams.

13.2. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a point load at the free end B is shown in Fig. 13.1. AB shows the position of cantilever before any load is applied whereas AB' shows the position of the cantilever after loading.

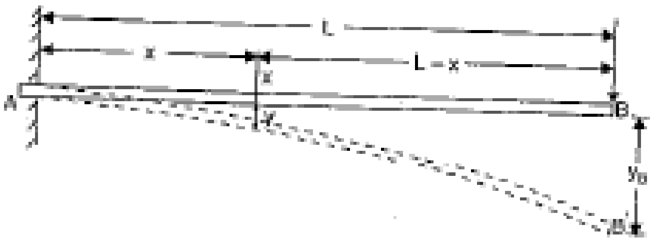


Fig. 13.1

Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -W(L-x) \quad \text{(Minus sign due to hogging)}$$

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = -W(L-x) = -WL + Wx$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \quad ...(i)$$

Integration again, we get

$$EIy = -WL\frac{x^2}{2} + \frac{W}{2}\frac{x^3}{3} + C_1x + C_2 \quad ...(ii)$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0$, $y = 0$ (*ii*) $x = 0$, $\frac{dy}{dx} = 0$

[At the fixed end, deflection and slopes are zero]

(*i*) By substituting $x = 0$, $y = 0$ in equation (*ii*), we get

$$0 = 0 + 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(*ii*) By substituting $x = 0$, $\frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = 0 + 0 + C_1 \quad \therefore \quad C_1 = 0$$

Substituting the value of C_1 in equation (*i*), we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2}$$

$$= -W\left(Lx - \frac{x^2}{2}\right) \quad ...(iii)$$

The equation (*iii*) is known as *slope equation*. We can find the slope at any point on the cantilever by substituting the value of x. The slope and deflection are maximum at the free end. These can be determined by substituting $x = L$ in these equations.

Substituting the values of C_1 and C_2 in equation (*ii*), we get

$$EIy = -WL\frac{x^2}{2} + \frac{Wx^3}{6} \quad (\because \quad C_1 = 0, C_2 = 0)$$

$$= -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \quad ...(iv)$$

The equation (*iv*) is known as *deflection equation*.

Let θ_B = slope at the free end B *i.e.*, $\left(\frac{dy}{dx}\right)$ at $B = \theta_B$ and

y_B = Deflection at the free end B

(*a*) Substituting θ_B for $\frac{dy}{dx}$ and $x = L$ in equation (*iii*), we get

$$EI.\theta_B = -W\left(L.L - \frac{L^2}{2}\right) = -W.\frac{L^2}{2}$$

$$\therefore \quad \theta_B = -\frac{WL^2}{2EI} \quad ...(13.1)$$

Negative sign shows that tangent at B makes an angle in the anti-clockwise direction with AB

$$\therefore \quad \theta_B = \frac{WL^2}{2EI} \quad ...(13.1A)$$

(b) Substituting y_B for y and $x = L$ in equation (iv), we get

$$EI.y_B = -W\left(L.\frac{L^2}{2} - \frac{L^3}{6}\right) = -W\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -W.\frac{L^3}{3}$$

$$\therefore \quad y_B = -\frac{WL^3}{3EI} \quad ...(13.2)$$

(Negative sign shows that deflection is downwards)

$$\therefore \quad \text{Downward deflection, } y_B = \frac{WL^3}{3EI} \quad ...(13.2\ A)$$

13.3. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point A and free at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig. 13.2.

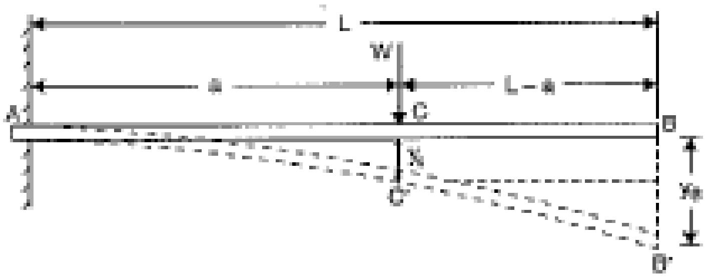


Fig. 13.2

Let θ_C = Slope at point C i.e., $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C

y_B = Deflection at point B

The portion AC of the cantilever may be taken as similar to a cantilever in Art. 13.1 (i.e., load at the free end).

$$\therefore \quad \theta_C = +\frac{Wa^2}{2EI} \quad \text{[In equation (13.1 A) change } L \text{ to } a\text{]}$$

and

$$y_C = \frac{Wa^3}{3EI} \quad \text{[In equation (13.2 A) change } L \text{ to } a\text{]}$$

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore

Slope at C = slope at B

or

$$\theta_C = \theta_B = \frac{Wa^2}{2EI} \quad ...(13.3)$$

Now from Fig. 13.2, we have

$$y_B = y_C + \theta_C(L - a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a) \quad \left(\because \theta_C = \frac{Wa^2}{2EI}\right) \quad ...(13.4)$$

Problem 13.1. *A cantilever of length 3 m is carrying a point load of 25 kN at the free end. If the moment of inertia of the beam = 10^8 mm^4 and value of E = 2.1 × 10^5 N/mm^2, find (i) slope of the cantilever at the free end and (ii) deflection at the free end.*

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

Point load, $W = 25$ kN $= 25000$ N

M.O.I., $I = 10^8$ mm^4

Value of $E = 2.1 \times 10^5$ N/mm^2

(*i*) Slope at the free end is given by equation (13.1 A).

$$\therefore \quad \theta_B = \frac{WL^2}{2EI} = \frac{25000 \times 3000^2}{2 \times 2.1 \times 10^5 \times 10^8} = \mathbf{0.005357\ rad.} \quad \textbf{Ans.}$$

(*ii*) Deflection at the free end is given by equation (13.2 A),

$$y_B = \frac{WL^3}{3EI} = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8} = \mathbf{10.71\ mm.} \quad \textbf{Ans.}$$

Problem 13.2. *A cantilever of length 3 m is carrying a point load of 50 kN at a distance of 2 m from the fixed end. If I = 10^8 mm^4 and E = 2 × 10^5 N/mm^2, find (i) slope at the free end and (ii) deflection at the free end.*

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

Point load, $W = 50$ kN $= 50000$ N

Distance between the load and the fixed end,

$a = 2$ m $= 2000$ mm

M.O.I., $I = 10^8$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

(*i*) Slope at the free end is given by equation (13.3) as

$$\theta_B = \frac{Wa^2}{2EI} = \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8} = \mathbf{0.005\ rad.} \quad \textbf{Ans.}$$

(*ii*) Deflection at the free end is given by equation (13.4) as

$$y_B = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a)$$

$$= \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000 - 2000)$$

$$= 6.67 + 5.0 = \mathbf{11.67\ mm.} \quad \textbf{Ans.}$$

13.4. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w per unit length over the whole length, is shown in Fig. 13.3.

Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -w(L - x) \cdot \frac{(L - x)}{2} \qquad \text{(Minus sign due to hogging)}$$

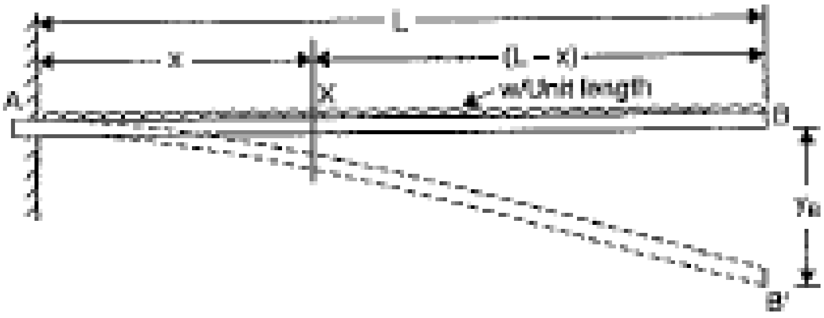


Fig. 13.3

But B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = -\frac{w}{2}\frac{(L-x)^3}{3}(-1) + C_1$$

$$= \frac{w}{6}(L-x)^3 + C_1 \qquad ...(i)$$

Integrating again, we get

$$EIy = \frac{w}{6}\cdot\frac{(L-x)^4}{4}(-1) + C_1x + C_2$$

$$= -\frac{w}{24}(L-x)^4 + C_1x + C_2 \qquad ...(ii)$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0$, $y = 0$ and (*ii*) at $x = 0$, $\frac{dy}{dx} = 0$ (as the deflection and slope at fixed end A are zero).

(*i*) By substituting $x = 0$, $y = 0$ in equation (*ii*), we get

$$0 = -\frac{w}{24}(L-0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$$

$$\therefore \qquad C_2 = \frac{wL^4}{24}$$

(*ii*) By substituting $x = 0$ and $\frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = \frac{w}{6}(L-0)^3 + C_1 = \frac{wL^3}{6} + C_1$$

$$\therefore \qquad C_1 = -\frac{wL^3}{6}$$

Substituting the values of C_1 and C_2 in equation (*i*) and (*ii*), we get

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \quad ...(iii)$$

and

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \quad ...(iv)$$

The equation (*iii*) is known as *slope equation* and equation (*iv*) as *deflection equation*. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point B, the value of $x = L$ is substituted in these equations.

Let θ_B = Slope at the free end B *i.e.*, $\left(\frac{dy}{dx}\right)$ at B

y_B = Deflection at the free end B.

From equation (*iii*), we get slope at B as

$$EI.\theta_B = \frac{w}{6}(L-L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$$

$$\therefore \quad \theta_B = -\frac{wL^3}{6EI} = -\frac{WL^2}{6EI} \quad (\because W = \text{Total load} = w.L) \quad ...(13.5)$$

From equation (*iv*), we get the deflection at B as

$$EI.y_B = -\frac{w}{24}(L-L)^4 - \frac{wL^3}{6}\times L + \frac{wL^4}{24}$$

$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$$

$$\therefore \quad y_B = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI} \quad (\because W = w.L)$$

$\therefore$ Downward deflection at B,

$$y_B = \frac{wL^4}{8EI} = \frac{WL^3}{8EI} \quad ...(13.6)$$

Problem 13.3. *A cantilever of length 2.5 m carries a uniformly distributed load of 16.4 kN per metre length over the entire length. If the moment of inertia of the beam = 7.95 × 10^7 mm^4 and value of $E = 2 \times 10^5$ N/mm^2, determine the deflection at the free end.*

Sol. Given :

Length, L = 2.5 mm = 2500 mm

U.d.l., w = 16.4 kN/m

$\therefore$ Total load, $W = w \times L = 16.4 \times 2.5 = 41$ kN = 41000 N

Value of $I = 7.95 \times 10^7$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

Let y_B = Deflection at the free end.

Using equation (13.6), we get

$$y_B = \frac{WL^3}{8EI} = \frac{41000 \times 2500^3}{8 \times 2 \times 10^5 \times 7.95 \times 10^7}$$

= **5.036 mm. Ans.**

Problem 13.7. *A cantilever of length 3 m carries two point loads of 2 kN at the free end and 4 kN at a distance of 1 m from the free end. Find the deflection at the free end.*

Take $E = 2 \times 10^5$ N/mm² and $I = 10^8$ mm⁴.

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

Load at free end, $W_1 = 2$ kN $= 2000$ N

Load at a distance one m from free end,

$W_2 = 4$ kN $= 4000$ N

Distance AC, $a = 2$ m $= 2000$ mm

Value of $E = 2 \times 10^5$ N/mm²

Value of $I = 10^8$ mm⁴

Let y_1 = Deflection at the free end due to load 2 kN alone

y_2 = Deflection at the free end due to load 4 kN alone.

Fig. 13.6

Downward deflection due to load 2 kN alone at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{2000 \times 3000^3}{3 \times 2 \times 10^5 \times 10^8} = 0.9 \text{ mm}.$$

Downward deflection at the free end due to load 4 kN (*i.e.*, 4000 N) alone at a distance 2 m from fixed end is given by (13.4) as

$$y_2 = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a)$$

$$= \frac{4000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{4000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000 - 2000)$$

$$= 0.54 + 0.40 = 0.94 \text{ mm}$$

∴ Total deflection at the free end

$$= y_1 + y_2$$

$$= 0.9 + 0.94 = \mathbf{1.84 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 13.8. *A cantilever of length 2 m carries a uniformly distributed load of 2.5 kN/m run for a length of 1.25 m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and $E = 1 \times 10^4$ N/mm².* (Annamalai University, 1990)

Sol. Given :

Length, $L = 2 \text{ m} = 2000 \text{ mm}$

U.d.l., $w = 2.5 \text{ kN/m} = 2.5 \times 1000 \text{ N/m}$

$$= \frac{2.5 \times 1000}{1000} \text{ N/mm} = 2.5 \text{ N/mm}$$

Point load at free end, $W = 1 \text{ kN} = 1000 \text{ N}$

Distance AC, $a = 1.25 \text{ m} = 1250 \text{ mm}$

Width, $b = 12 \text{ mm}$

Depth, $d = 24 \text{ mm}$

Value of $I = \frac{bd^3}{12} = \frac{12 \times 24^3}{12}$

$$= 13824 \text{ cm}^4 = 13824 \times 10^4 \text{ mm}^4 = 1.3824 \times 10^8 \text{ mm}^4$$

Value of $E = 1 \times 10^4 \text{ N/mm}^2$

Let y_1 = Deflection at the free end due to point load 1 kN alone

y_2 = Deflection at the free end due to u.d.l. on length AC.

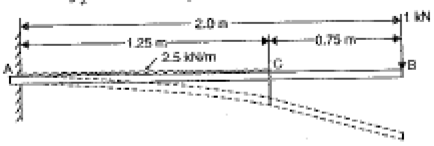


Fig. 13.7

(*i*) Now the downward deflection at the free end due to point load of 1 kN (or 1000 N) at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{1000 \times 2000^3}{3 \times 10^4 \times 1.3824 \times 10^8} = 1.929 \text{ mm.}$$

(*ii*) The downward deflection at the free end due to uniformly distributed load of 2.5 N/mm on a length of 1.25 m (or 1250 mm) is given by equation (13.8) as

$$y_2 = \frac{wa^4}{8EI} + \frac{w \cdot a^3}{6EI}(L - a)$$

$$= \frac{2.5 \times 1250^4}{8 \times 10^4 \times 1.3824 \times 10^8} + \frac{2.5 \times 1250^3}{6 \times 10^4 \times 1.3824 \times 10^8} (2000 - 1250)$$

$$= 0.5519 + 0.4415 = 0.9934$$

∴ Total deflection at the free end due to point load and u.d.l.

$$= y_1 + y_2 = 1.929 + 0.9934 = \mathbf{2.9224\ mm.} \quad \textbf{Ans.}$$

Problem 13.9. *A cantilever of length 2 m carries a uniformly distributed load 2 kN/m over a length of 1 m from the free end, and a point load of 1 kN at the free end. Find the slope and deflection at the free end if $E = 2.1 \times 10^5$ N/mm² and $I = 6.667 \times 10^7$ mm⁴.*

Sol. Given : (See Fig. 13.8)

Length, $L = 2$ m $= 2000$ mm

U.d.l. $w = 2$ kN/m $= \frac{2 \times 1000}{1000}$ N/mm $= 2$ N/mm

Length BC, $a = 1$ m $= 1000$ mm

Point load, $W = 1$ kN $= 1000$ N

Value of $E = 2.1 \times 10^5$ N/mm²

Value of $I = 6.667 \times 10^7$ mm⁴.

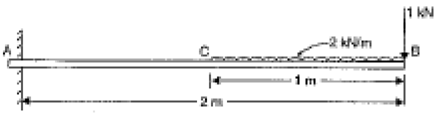


Fig. 13.8

(i) Slope at the free end

Let θ_1 = Slope at the free end due to point load of 1 kN *i.e.*, 1000 N

θ_2 = Slope at the free end due to u.d.l. on length BC.

The slope at the free end due to a point load of 1000 N at B is given by equation (13.1 A) as

$$\theta_1 = \frac{WL^2}{2EI} \qquad (\because \ \theta_B = \theta_1 \text{ here})$$

$$= \frac{1000 \times 2000^2}{2 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.0001428 \text{ rad.}$$

The slope at the free end due to u.d.l. of 2 kN/m over a length of 1 m from the free end is given by equation (13.9) as

$$\theta_2 = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI} \qquad (\because \ \theta_B = \theta_2 \text{ here})$$

$$= \frac{2 \times 2000^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000 - 1000)^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}$$

$$= 0.0001904 - 0.000238 = 0.0001666 \text{ rad.}$$

∴ Total slope at the free end

$$= \theta_1 + \theta_2 = 0.0001428 + 0.0001666 = \mathbf{0.0003094\ rad.} \quad \textbf{Ans.}$$

(ii) Deflection at the free end

Let y_1 = Deflection at the free end due to point load of 1000 N

y_2 = Deflection at the free end due to u.d.l. on length BC.

The deflection at the free end due to point load of 1000 N is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} \qquad (\because \ \text{Here } y_1 = y_B)$$

$$= \frac{1000 \times 2000^3}{3 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.1904 \text{ mm.}$$

The deflection at the free end due to u.d.l. of 2 N/mm over a length of 1 m from the free end is given by equation (13.10) as

$$y_2 = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]$$

$$= \frac{2 \times 2000^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \left[\frac{2(2000 - 1000)^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} + \frac{2(2000 - 1000)^3 \times 1000}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}\right]$$

$$= 0.2857 - [0.01785 + 0.0238] = 0.244 \text{ mm}$$

∴ Total deflection at the free end

$$= y_1 + y_2 = 0.1904 + 0.244 = \mathbf{0.4344\ mm.} \quad \textbf{Ans.}$$