Type-2 : Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \{\ldots\ x(-2) z^{2} + x(-1) z^{1} + x(0) + x(1) z^{-1} + x(2)(z^{-2}) + \ldots\}$$

$$x(n) = \{\ldots\ x(-2), x(-1), x(0), x(1), x(2), \ldots\}$$

$x(n)$ can be obtained by collecting the coefficient of z in $X(z)$ expansion.

Note:

* When the ROC is $|z| > |a|$ [Causal system] then expand $X(z)$ such that the power of z are negative.
* When the ROC is $|z| < |a|$ [Non-causal system] then expand $X(z)$ such that the power of z are positive.

Q) Find the inverse z-transform using power series expansion.

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC } |z| > |a|$$
$$|z| < |a|$$

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{\overset{(-)}{1} \; \overset{(+)}{-}\, az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{\overset{(-)}{az^{-1}} \; \overset{(+)}{-}\, a^2z^{-2}} \\
 & \qquad a^2z^{-2} \\
 & \qquad \underline{\overset{(-)}{a^2z^{-2}} \; \overset{(+)}{-}\, a^3z^{-3}} \\
 & \qquad\qquad a^3z^{-3} \dots
\end{array}
$$

$x(z) = \{1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots\}$

$x(n) = \{1, a, a^2, a^3, \dots\}$

$\therefore$ Causal system $\quad x(n) = a^n u(n)$

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 \\
\hline
az^{-1} + 1 & 1 \\
 & \underline{\overset{(-)}{1} \; \overset{(+)}{-}\, a^{-1}z} \\
 & \quad a^{-1}z \\
 & \quad \underline{\overset{(-)}{a^{-1}z} \; \overset{(+)}{-}\, a^{-2}z^2} \\
 & \qquad a^{-2}z^2 \\
 & \qquad \underline{\overset{(-)}{a^{-2}z^2} \; \overset{(+)}{-}\, a^{-3}z^3} \\
 & \qquad\qquad a^{-3}z^3
\end{array}
$$

$X(z) = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3$

$X(z) = \{\dots\dots -a^{-3}z^3 - a^{-2}z^2 - a^{-1}z\}$

$X(n) = \{\dots, -a^{-3}, -a^{-2}, -a^{-1}\}$ $\therefore$ Non-Causal System

HW

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

when $x(n)$ is causal & non-causal.

$$
\begin{array}{r|l}
 & 1+4z^{-1}+7z^{-2}+10z^{-3} \\
\hline
1-2z^{-1}+z^{-2} & 1+2z^{-1} \\
 & \underset{(-)}{1} \; \underset{(+)}{-2z^{-1}} \; \underset{(-)}{+z^{-2}} \\
\hline
 & 4z^{-1} - z^{-2} \\
 & \underset{(-)}{4z^{-1}} \; \underset{(+)}{-8z^{-2}} \; \underset{(-)}{+4z^{-3}} \\
\hline
 & \cancel{7z^{-2}} - 4z^{-3} \\
 & \underset{(-)}{\cancel{7z^{-2}}} \; \underset{(+)}{-14z^{-3}} \; \underset{(-)}{+7z^{-4}} \\
\hline
 & 10z^{-3} - 7z^{-4}
\end{array}
$$

$$X(z) = 1+4z^{-1}+7z^{-2}+10z^{-3}$$

$$x(n) = \{1, 4, 7, 10, \dots\}$$

$$
\begin{array}{r|l}
 & 2z+5z^{2}+8z^{3}+11z^{4} \\
\hline
z^{-2}-2z^{-1}+1 & 2z^{-1}+1 \\
 & \underset{(-)}{2z^{-1}} \; \underset{(+)}{-4} \; \underset{(-)}{+2z} \\
\hline
 & \cancel{5} - 2z \\
 & \underset{(-)}{\cancel{5}} \; \underset{(+)}{-10z} \; \underset{(-)}{+5z^{2}} \\
\hline
 & 8z - 5z^{2} \\
 & \underset{(-)}{8z} \; \underset{(+)}{-16z^{2}} \; \underset{(-)}{+8z^{3}} \\
\hline
 & 11z^{2} - 8z^{3}
\end{array}
$$

$$x(Z) = \{ \ldots\ldots 11z^4 + 8z^3 + 5z^2 + 2Z \}$$

$$x(n) = \{ + 11, 8, 5, 2, 0, \ldots \}$$

Q) $X(Z) = \dfrac{Z+1}{Z^2 - 3Z + 2}$

i) $x(n)$ is causal

ii) $x(n)$ is non-causal

$$Z^{-1} + 4Z^{-2} + 10Z^{-3} + 22Z^{-4}$$

$Z^2 - 3Z + 2 \;\big|\; Z + 1$

$\cancel{Z} - 3 + 2Z^{-1}$

$(-) \quad (+) \quad (-)$

$\cancel{4} - 2Z^{-1}$

$\cancel{4} - 12Z^{-1} + 8Z^{-2}$

$(-) \quad (+) \quad (-)$

$\cancel{10Z^{-1}} - 8Z^{-2}$

$\cancel{10Z^{-1}} - 30Z^{-2} + 20Z^{-3}$

$(-) \quad (+) \quad (-)$

$\cancel{22Z^{-2}} - 20Z^{-3}$

$\cancel{22Z^{-2}} - 66Z^{-3} + 44Z^{-4}$

$(-) \quad (+) \quad (-)$

$46Z^{-3} - 44Z^{-4}$

$$x(Z) = \{ Z^{-1} + 4Z^{-2} + 10Z^{-3} + 22Z^{-4} + \ldots \}$$

$$x(n) = \{ 0, 1, 4, 10, 22, \ldots\ldots \}$$

$\therefore$ $x(n)$ is causal.

ii)

$$\frac{1}{2} + \frac{5}{4}z + \frac{13}{8}z^2 + \cdots\cdots$$

$$2 - 3z + z^2 \,\big)\, 1 + z$$

$$1 - \frac{3}{2}z + \frac{1}{2}z^2$$

$$\frac{5}{2}z - \frac{1}{2}z^2$$

$$\frac{5}{2}z - \frac{15}{4}z^2 + \frac{5}{4}z^3$$

$$\frac{13}{4}z^2 - \frac{5}{4}z^3$$

$$\frac{13}{4}z^2 - \frac{39}{8}z^3 + \frac{13}{8}z^4$$

$$\frac{2\cancel{4}}{8}z^3 - \frac{13}{8}z^4$$