Z - TRANSFORM

Defn: z - transform [Two sided (or) bilateral]

Let $\{f(n)\}$ be a sequence defined for all integers then its z - transform is defined to be

$$F(z) = Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}.$$

where z is an arbitrary complex number.

Defn: z - transform [One - sided (or) unilateral]

Let $\{f(n)\}$ be a sequence defined for all positive integers then the z - transform of $\{f(n)\}$ is defined to be

$$F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

Defn: z - transform for discrete values of t.

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, \ldots$ T being the sampling period, then z - transform of $f(t)$ is defined as $F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$.

P.T. $Z(1) = \frac{z}{z-1}, \ |z|>1$.

We know $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z(1) = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \qquad \left[(1-x)^{-1} = 1 + x + x^2 + \cdots\right]$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$Z(1) = \frac{z}{z-1}$$

② Find $Z(-1)^n$.

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z(-1)^n = \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n$$

$$= 1 - \frac{1}{z} + \left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3 + \cdots$$

$$= 1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots$$

$$= \left(1 + \frac{1}{z}\right)^{-1} \qquad \left[(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \right.$$

$$z(-1)^n = \frac{z}{z+1}$$

(3) Find $z(a^n)$

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

$$z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}.$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n.$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} \quad \left[\because (1-x)^{-1} = 1 + x + x^2 + \cdots \right]$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$z(a^n) = \frac{z}{z-a}.$$

(4) Find $z(n)$.

$$z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

$$z(n) = \sum_{n=0}^{\infty} n z^{-n}.$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n} = \sum_{n=0} n \cdot \left(\frac{1}{z}\right)^n.$$

$$= 0 + \frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= \frac{1}{z}\left[\left(1 - \frac{1}{z}\right)^{-2}\right] \quad \left[\because (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots\right]$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left(\frac{z}{z-1}\right)^2 = \frac{1}{\cancel{z}} \cdot \frac{z^{\cancel{2}}}{(z-1)^2}$$

$$z(n) = \frac{z}{(z-1)^2}$$

Find $z\left(\frac{1}{n}\right)$, $n \neq 0$

$$z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

$$z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} \cdot z^{-n} \quad \left[\because n \neq 0\right]$$

$$= \sum_{n=1}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n}$$

$$= \frac{(1/z)^1}{1} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \cdots$$

$$= \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots$$

$$= -\log\left(1-\frac{1}{z}\right) = \log\left(1-\frac{1}{z}\right)^{-1} = \log\left(\frac{z-1}{z}\right)^{-1}$$

$$z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$$

7) Find $z\left(\frac{1}{n+1}\right)$

$$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n+1}$$

$$= \frac{(1/z)^0}{1} + \frac{(1/z)^1}{2} + \frac{(1/z)^2}{3} + \cdots$$

Multiply & divide by $1/z$.

$$= \left(\frac{1}{z}\right)^{-1}\left[1/z + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \cdots\right]$$

$$= \left(\frac{1}{z}\right)^{-1}\left[-\log(1-1/z)\right] = \left(\frac{1}{z}\right)^{-1}\log\left[\frac{z-1}{z}\right]^{-1}$$

$$= z\log\left[\frac{z}{z-1}\right]$$