Solution of Two Dimensional Heat Flow Equation

The two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

The possible solutions of two dimensional heat equation is

(i) $u(x,y)=(Ae^{px}+Be^{-px})(C\cos py+D\sin py)$

(ii) $u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py})$

(iii) $u(x,y)=(Ax+B)(Cy+D)$

The suitable soln. is TYPE-I Heat flows in x direction (i.e) $0<x<l$

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}).$$

The boundary condtns. are:

i) $u(0,y)=0$.

ii) $u(l,y)=0$

iii) $u(x,0)=0$.

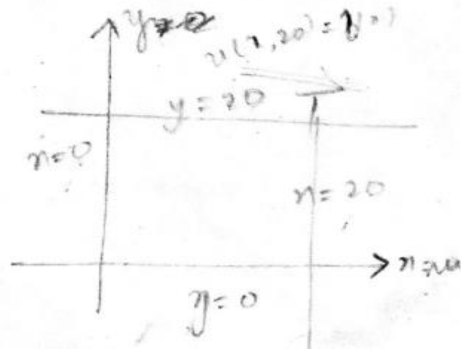
iv) $u(x,l)=f(x)$. $0<x<l$.

1 A square plate is bdd. by the lines $x=0, y=0$, $x=20$ and $y=20$. Its faces are insulated. The temp. along the upper horizontal edge is gn. by $u(x,20)=x(20-x)$ when $0<x<20$ while the other three edges are kept at $0°C$. Find the steady state temp. in the plate .

Soln: Let $u(x,y)$ be the temp. at any point (x,y).

Then $u(x,y)$ satisfies the Laplace's eqn.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



The boundary condns. are:

(i) $u(0,y)=0$

(ii) $u(20,y)=0$

(iii) $u(x,0)=0$

(iv) $u(x,20)=x(20-x),\ 0<x<20$.

The suitable soln. is

$$\cancel{u(x,y)=(Ae^{px}+Be^{-px})C}$$

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}) \quad \text{——} (1)$$

Apply (i) in (1)

$$u(0,y)=A(Ce^{py}+De^{-py})$$

$$0=A(Ce^{py}+De^{-py}) \Rightarrow \boxed{A=0}$$

$$\therefore u(x,y)=B\sin px\,(Ce^{py}+De^{-py}) \quad \text{——} (2)$$

Apply (ii) in ②

$$u(20,y) = B\sin 20p\,(Ce^{py} + De^{-py})$$

$$0 = B\sin 20p\,(Ce^{py} + De^{-py})$$

$$\Rightarrow B \neq 0,\quad \sin 20p = 0$$

$$\sin 20p = \sin n\pi$$

$$\boxed{p = \frac{n\pi}{20}}$$

$$\therefore u(x,y) = B\sin \frac{n\pi}{20}x \left(Ce^{\frac{n\pi}{20}y} + De^{-\frac{n\pi}{20}y}\right) \quad \text{—— ③}$$

Apply (iii) in ③

$$u(x,0) = B\sin \frac{n\pi}{20}x\,(C+D)$$

$$0 = B\sin \frac{n\pi}{20}x\,(C+D)$$

$$\Rightarrow C+D = 0$$

$$\Rightarrow \boxed{D = -C}$$

$$\therefore u(x,y) = B\sin \frac{n\pi}{20}x \left(Ce^{\frac{n\pi}{20}y} - Ce^{-\frac{n\pi}{20}y}\right)$$

$$= BC\sin \frac{n\pi}{20}x \left(e^{\frac{n\pi}{20}y} - e^{-\frac{n\pi}{20}y}\right)$$

$$= BC\sin \frac{n\pi}{20}x \left(2\sinh \frac{n\pi y}{20}\right)$$

$$u(x,y) = 2BC\sin \frac{n\pi}{20}x \sinh \frac{n\pi y}{20}$$

$\therefore$ The general soln. is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{20} x \sinh \frac{n\pi y}{20} \quad \text{--- (4)}$$

Apply (iv) in (4)

$$u(x,20) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{20} x \sinh \frac{n\pi}{20} \cdot 20$$

$$= \sum_{n=1}^{\infty} A_n \sinh n\pi \sin \frac{n\pi x}{20}$$

$$x(20-x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20} \quad \text{where } B_n = A_n \sinh n\pi.$$

$$B_n = \frac{2}{20} \int_0^{20} x(20-x) \sin \frac{n\pi x}{20} \, dx$$

$$= \frac{1}{10} \int_0^{20} (20x - x^2) \sin \frac{n\pi x}{20} \, dx.$$

$$= \frac{1}{10} \left[20x \left(-\cos \frac{n\pi x}{20} \right) \cdot \frac{20}{n\pi} - 20 \left(-\sin \frac{n\pi x}{20} \right) \left(\frac{20}{n\pi} \right)^2 \right]_0^{20}$$

$$- \frac{1}{10} \left[x^2 \left(-\cos \frac{n\pi}{20} x \right) \cdot \frac{20}{n\pi} - 2x \left(-\sin \frac{n\pi}{20} x \right) \left(\frac{20}{n\pi} \right)^2 + 2 \left(\cos \frac{n\pi}{20} x \right) \left(\frac{20}{n\pi} \right)^3 \right]_0^{20}$$

$$= \frac{1}{10}\left[-400(-1)^n \frac{20}{n\pi} + 0\right] - \frac{1}{10}\left[400(-1)^n \frac{20}{n\pi} + 2(-1)^n\left(\frac{20}{n\pi}\right)^3 - 2\left(\frac{20}{n\pi}\right)^3\right]$$

$$= \frac{1}{10}\left[\cancel{-400(-1)^n \frac{20}{n\pi}} + \cancel{400(-1)^n \frac{20}{n\pi}} - 2(-1)^n \frac{(20)^3}{(n\pi)^3} + 2\left(\frac{20}{n\pi}\right)^3\right]$$

$$= \frac{1}{5}\left[1-(-1)^n\right]\left(\frac{20}{n\pi}\right)^3$$

$$A_n = \frac{B_n}{\sinh n\pi} = \frac{1}{5}\frac{\left[1-(-1)^n\right]}{\sinh n\pi}\left(\frac{20}{n\pi}\right)^3$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{1}{\cancel{5}}\frac{\left[1-(-1)^n\right]}{\sinh n\pi} \cdot \frac{\overset{1600}{\cancel{8000}}}{n^3\pi^3} \sinh\frac{n\pi y}{20} \sin\frac{n\pi x}{20}$$

$$= \sum_{n=1}^{\infty} \frac{1600}{n^3\pi^3} \cdot \frac{\left[1-(-1)^n\right]}{\sinh n\pi} \cdot \sinh\frac{n\pi y}{20} \cdot \sin\frac{n\pi x}{20}$$