HW: A tightly streched string with fixed end pts. $x=0$ & $x=l$ is initially in a position gn. by $y(x,0)=y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from the position, find the displacement y at any time and at any distance from the end $x=0$

Soln: The general form of one dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{— (1)}$$

Suitable soln. for one dimensional wave eqn is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad \text{— (2)}$$

Boundary conditions:

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$ &

(iv) $y(x,0)$

By cdtn. (i) in (2) we have,

$$y(0,t) = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D.\sin pat)$$

$$\Rightarrow \boxed{A = 0}$$

Sub $A = 0$ in (2)

$$y(x,t) = B \sin px \,(C \cos pat + D \sin pat) \rightarrow (3)$$

By cdtn. (ii) in (3) we have,

$$y(l,t) = B \sin pl \,(C \cos pat + D \sin pat)$$

$$0 = B \sin pl \,(C \cos pat + D \sin pat)$$

$$B \sin pl = 0$$

If $B = 0$; suitable soln. is zero

If $B \neq 0$; $\sin pl = 0$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Sub $p = \frac{n\pi}{l}$ in (3)

$$y(x,t) = B \sin \frac{n\pi}{l}x \left[C \cos \frac{n\pi}{l}at + D \sin \frac{n\pi}{l}at\right] \rightarrow (4)$$

By cdtn (iii) in (4),

w.r.t. t
$$\frac{\partial y}{\partial t}(x,t) = B \sin \frac{n\pi x}{l}\left[-C \sin \frac{n\pi}{l}at \cdot \left(\frac{n\pi a}{l}\right) + D \cos \frac{n\pi}{l}at \cdot \left(\frac{n\pi a}{l}\right)\right]$$

$$= B\sin\frac{n\pi x}{l}\cdot\left(\frac{n\pi a}{l}\right)\left[-C\sin\frac{n\pi a}{l}t + D\cos\frac{n\pi a}{l}t\right]$$

$$\frac{\partial y}{\partial t}(x,0) = B\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l}\left[-C\sin\frac{n\pi a}{l}(0) + D\cos\frac{n\pi a}{l}(0)\right]$$

$$= B\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l}[D]$$

$$\frac{\partial y}{\partial t}(x,0) = BD\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l}$$

$$0 = BD\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l} \Rightarrow D = 0$$

Sub $D = 0$ in (4)

$$y(x,t) = \left(B\sin\frac{n\pi x}{l}\right)\left(C\cos\frac{n\pi}{l}at\right) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi}{l}at$$

General soln. is

$$y(x,t) = \sum_{n=1}^{\infty} B\sin\frac{n\pi x}{l}\cos\frac{n\pi}{l}at$$

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi a}{l}t \quad \text{---} (5) \quad \boxed{BC = B_n}$$

By cdtn. (iv) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}$$

$$y = k(lx - x^2) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{l}x \quad \text{---} (6)$$

Apply (iv) we get,

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x.$$

$$y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$$

$$\left\{ \sin 3\theta = 3\sin\theta - 4\sin^3\theta \right.$$

$$\cancel{\sin^3\theta = \sin 3\theta - 3\sin\theta}$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$\frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}.$$

$$\frac{y_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\frac{3\pi x}{l}\right] = B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + \cdots$$

Equating like terms on BS we get,

$$B_1 = \frac{3y_0}{4}, \quad B_2 = 0, \quad B_3 = -\frac{y_0}{4}, \ldots B_4 = 0, \ldots$$

$$\therefore B_1 = \frac{3y_0}{4} \ \& \ B_3 = -\frac{y_0}{4}, \quad B_n = 0 \text{ for } n \neq 1, 3.$$

$$\therefore y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}.$$