

$$\text{Centre} = (-g, -f) \\ = \left(-\frac{5}{4}, \frac{1}{2}\right)$$

$$r = \sqrt{g^2 + f^2 - c} \\ = \sqrt{\frac{25}{16} + \frac{1}{4} - \frac{1}{2}} \\ = \sqrt{\frac{25 + 4 - 8}{16}} \\ = \sqrt{\frac{21}{16}} \\ = \frac{\sqrt{21}}{4}$$

$$\text{Radius of curvature } \rho = \frac{\sqrt{21}}{4}$$

$$\text{Curvature} = \frac{1}{\rho}$$

$$= \frac{4}{\sqrt{21}}$$

Example 3

Find the radius of curvature $xy = c^2$ at $x = c$

$$xy = c^2 \\ y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = c^2 \left(-\frac{1}{x^2}\right)$$

$$= -\frac{c^2}{x^2} \quad \frac{dy}{dx}(c, c) = -\frac{c^2}{c^2} = -1$$

Given $x = c$

$$y_1 = \frac{c^2}{c^2} = 1$$

$$y = \frac{c^2}{c} = c$$

$$\frac{d^2y}{dx^2} = -c^2 \left(\frac{-2}{x^3}\right)$$

$$= \frac{2c^2}{x^3}$$

$$\frac{d^2y}{dx^2}(c, c) = \frac{2c^2}{c^3} = \frac{2}{c}$$

$$\begin{aligned}
 \text{Radius of curvature } \rho &= \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} \\
 &= \frac{\left[1 + (-1)^2\right]^{3/2}}{2/e} \\
 &= \frac{(1+1)^{3/2}}{2/e} \\
 &= \frac{(\sqrt{2})^3}{2/e} \\
 &= \frac{2\sqrt{2}e}{2} = \sqrt{2}e
 \end{aligned}$$

4. Find the radius of curvature $y=e^x$ at $(0,1)$
Sol given $y=e^x$

$$\frac{dy}{dx} = e^x, \text{ at } (0,1), \frac{dy}{dx} \Big|_{(0,1)} = e^0 = 1$$

$$\frac{d^2y}{dx^2} = e^x, \text{ at } (0,1), \frac{d^2y}{dx^2} \Big|_{(0,1)} = e^0 = 1$$

$$\begin{aligned}
 \rho &= \frac{\left(1 + (y_1)'^2\right)^{3/2}}{y_2} \\
 &= \frac{(1+1)^{3/2}}{1} \\
 &= 2^{3/2} = 2\sqrt{2}
 \end{aligned}$$

5. Find the radius of curvature $y = c \cos h\left(\frac{x}{c}\right)$ at $(0, c)$

$$\frac{dy}{dx} = c \sin h\left(\frac{x}{c}\right) \cdot \frac{1}{c}$$

$$= \sin h\left(\frac{x}{c}\right)$$

$$\begin{aligned}
 \frac{dy}{dx} \Big|_{(0,c)} &= \sin h\left(\frac{0}{c}\right) \\
 &= 0
 \end{aligned}$$

$$\frac{d^2y}{dx^2} = \cos h\left(\frac{x}{c}\right) \frac{1}{c}$$

$$= \cos h\left(\frac{0}{c}\right) \frac{1}{c}$$

$$= 1 \times \frac{1}{c} = \frac{1}{c}$$

$$p = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+0)^{3/2}}{1/c}$$

$$p = c$$

b. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 1$ at $(1/4, 1/4)$

$$\sqrt{x} + \sqrt{y} = 1$$

Diff w.r. to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y_1 = 0 \rightarrow \textcircled{1}$$

$$y_1 = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\frac{\sqrt{y}}{\sqrt{x}} \rightarrow \textcircled{1}$$

$$y_1(1/4, 1/4) = -\frac{\sqrt{1/4}}{\sqrt{1/4}} = -1$$

Diff $\textcircled{1}$ w.r. to

$$d(uv) = \frac{vu' - uv'}{v^2} \quad u = \sqrt{y}, \quad v = \sqrt{x}$$

$$u' = \frac{1}{2\sqrt{y}} y_1, \quad v' = \frac{1}{2\sqrt{x}}$$

$$y_2 = -\left[\frac{\sqrt{x} \frac{1}{2\sqrt{y}} y_1 - \sqrt{y} \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \right]$$

$$y_2(1/4, 1/4) = -\left[\frac{\sqrt{1/4} \frac{1}{2\sqrt{1/4}} y_1(-1) - \sqrt{1/4} \frac{1}{2\sqrt{1/4}}}{(1/4)} \right]$$

$$= -\left[\frac{-\frac{1}{2} - \frac{1}{2}}{1/4} \right] = \frac{1/2 + 1/2}{1/4}$$

$$= 4$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_1}$$

$$= \frac{(1+1)^{3/2}}{4}$$

$$= \frac{2^{3/2}}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\rho = \frac{1}{\sqrt{2}}$$

Find ρ where $x^3+y^3=3axy$ at $(\frac{3a}{2}, \frac{3a}{2})$

Given $x^3+y^3=3axy$

Points $(\frac{3a}{2}, \frac{3a}{2})$

Diff w.r. to x

$$3x^2 + 3y^2 \cdot y_1 = 3a [xy_1 + y]$$

$$y^2 \cdot y_1 = ay_1 + ay - x^2$$

$$y_1 (y^2 - ax) = ay - x^2$$

$$y_1 = \frac{ay - x^2}{y^2 - ax} \rightarrow \textcircled{1}$$

$$y_1 \left(\frac{3a}{2}, \frac{3a}{2} \right) = \frac{a \left(\frac{3a}{2} \right) - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \left(\frac{3a}{2} \right)}$$

$$= \frac{3a^2/2 - 9a^2/4}{9a^2/4 - 3a^2/2}$$

$$= - \left[\frac{9a^2/4 - 3a^2/2}{9a^2/4 - 3a^2/2} \right]$$

$$= -1$$

Diff $\textcircled{1}$ w.r. to x

$$y_2 = \frac{uv' - u'v}{v^2}$$

$$u = ay - x^2 \quad v = y^2 - ax$$

$$u' = ay_1 - 2x \quad v' = 2yy_1 - a$$

$$y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right) \left(-a - 2 \cdot \frac{3a}{2}\right) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right) \left(-2 \cdot \frac{3a}{2} - a\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-4a) + \left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-4a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-4a - 4a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$\frac{-8a}{\frac{9a^2 - 6a^2}{4}} = \frac{-32a}{3a^2}$$

$$= \frac{-32}{3a}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{-32/3a}$$

$$= \frac{(2)^{3/2}}{-32/3a} = \frac{2\sqrt{2} \times 3a}{-32}$$

$$\rho = -\frac{3a\sqrt{2}}{16}$$

$$\rho = +\frac{3a\sqrt{2}}{16} \quad (\rho \text{ is always +ve})$$

8 Find radius of curvature ρ , at $xy = 12$ at $(3, 4)$

$$xy = 12$$

$$y = \frac{12}{x}$$

Diff w.r. to x

$$y_1 = 12 \left(-\frac{1}{x^2} \right)$$

$$= -\frac{12}{x^2} \rightarrow \textcircled{1}$$

$$y_1(3, 4) = -\frac{12}{9}$$

$$= -\frac{4}{3}$$

Diff $\textcircled{1}$ w.r. to x

$$y_2 = -12 \left(\frac{-2}{x^3} \right)$$

$$= \frac{24}{x^3}$$

$$y_2(3, 4) = \frac{24}{27}$$

$$= \frac{8}{9}$$

The radius of curvature $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{[1 + (-\frac{4}{3})^2]^{3/2}}{\frac{8}{9}}$$

$$= \frac{\frac{81}{9} (1 + \frac{16}{9})^{3/2}}{\frac{8}{9}} = \frac{(25/9)^{3/2}}{\frac{8}{9}}$$

$$= \frac{25\sqrt{25}}{9\sqrt{9}} \times \frac{9}{8}$$

$$= \frac{25 \times 5}{9 \times 3} \times \frac{9}{8}$$

$$= \frac{125}{24}$$