ISOPARAMETRIC FORMULATION

Session objectives

At the end of this session, the learner will be able to

1) understand 2D **ISOPARAMETRIC** problems in engineering
2) understand and analyze Natural co-ordinate systemsproblems

Learning Outcome

Students should be able to

1) understand 2D **ISOPARAMETRIC** problems in engineering
2) understand and analyze Natural co-ordinate systemsproblems

Teaching learning material

- Board/White Board and Markers
- Presentation/PPT

ISOPARAMETRIC FORMULATION

Element Equation For Four - Noded Quadrilateral Element Using Natural Coordinates

Strain - Displacement matrix equation

We know that the strain - displacement relations for a two dimensional element is given by

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

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The displacement function for parent rectangular element is given by

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

where N_1, N_2, N_3 and N_4 are shape functions. Also due to isoparametric characteristics, the coordinates of general quadrilateral element can be expressed by the same shape functions.

i.e.,
$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}$$

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We have to express the derivatives of function in x, y coordinates in terms of its derivatives in ε, η coordinates, since the shape functions N_1 to N_4 are the function of ε and η.

Hence to derive for element strains (i.e., $\frac{\partial u}{\partial x}\ \frac{\partial v}{\partial y}$ etc) we must use the chain rule of differentiation.

$$\Rightarrow \quad \frac{\partial u}{\partial \varepsilon} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \varepsilon} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \varepsilon}$$

$$\text{and} \quad \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}$$

Arranging the above equations in matrix form

$$\Rightarrow \quad \begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \varepsilon} & \frac{\partial y}{\partial \varepsilon} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix}$$

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$$\Rightarrow \begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix}$$

Where J is the Jacobian matrix

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \varepsilon} & \frac{\partial y}{\partial \varepsilon} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where, $J_{11} = \frac{\partial x}{\partial \varepsilon}$; $J_{12} = \frac{\partial y}{\partial \varepsilon}$

$J_{21} = \frac{\partial x}{\partial \eta}$; $J_{22} = \frac{\partial y}{\partial \eta}$

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From equation

$$\Rightarrow \quad x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

$$J_{11} = \frac{\partial x}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon}x_1 + \frac{\partial N_2}{\partial \varepsilon}x_2 + \frac{\partial N_3}{\partial \varepsilon}x_3 + \frac{\partial N_4}{\partial \varepsilon}x_4$$

$$J_{12} = \frac{\partial y}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon}y_1 + \frac{\partial N_2}{\partial \varepsilon}y_2 + \frac{\partial N_3}{\partial \varepsilon}y_3 + \frac{\partial N_4}{\partial \varepsilon}y_4$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta}x_1 + \frac{\partial N_2}{\partial \eta}x_2 + \frac{\partial N_3}{\partial \eta}x_3 + \frac{\partial N_4}{\partial \eta}x_4$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \eta}y_1 + \frac{\partial N_2}{\partial \eta}y_2 + \frac{\partial N_3}{\partial \eta}y_3 + \frac{\partial N_4}{\partial \eta}y_4$$

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We know that the shape functions are given by

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

$$\Rightarrow \quad \frac{\partial N_1}{\partial \varepsilon} = \frac{1}{4}(-1)\times(1-\eta) = \frac{1}{4}\times-(1-\eta)$$

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$$\frac{\partial N_2}{\partial \varepsilon} = \frac{1}{4}(1)\times(1-\eta) = \frac{1}{4}\times(1-\eta)$$

$$\frac{\partial N_3}{\partial \varepsilon} = \frac{1}{4}(1)\times(1+\eta) = \frac{1}{4}\times(1+\eta)$$

$$\frac{\partial N_4}{\partial \varepsilon} = \frac{1}{4}(-1)\times(1+\eta) = \frac{1}{4}\times-(1+\eta)$$

$$\frac{\partial N_1}{\partial \eta} = \frac{1}{4}(1-\varepsilon)(-1) = \frac{1}{4}\times-(1-\varepsilon)$$

$$\frac{\partial N_2}{\partial \eta} = \frac{1}{4}(1+\varepsilon)(-1) = \frac{1}{4}\times-(1+\varepsilon)$$

$$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\varepsilon)(1) = \frac{1}{4}\times(1+\varepsilon)$$

$$\frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\varepsilon)(1) = \frac{1}{4}\times(1-\varepsilon)$$

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Substitute above values in equations

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4]$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4]$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4]$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 + (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4]$$

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Substituting above values in equations and writting in matrix form as

$$[J] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\varepsilon) & -(1+\varepsilon) & (1+\varepsilon) & (1-\varepsilon) \end{bmatrix} \times \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

From equation , we know that

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

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i.e.,

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \quad \ldots$$

where $|J|$ is the determinant of Jacobian matrix.

Following the same procedure for displacement v, we can write

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{Bmatrix} \frac{\partial v}{\partial \varepsilon} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} \quad \ldots$$

Now equations *Of* yields

$$\text{strain} \quad \{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

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$$= \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \varepsilon} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

We know that,

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

$$\Rightarrow \quad \frac{\partial u}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon}u_1 + \frac{\partial N_2}{\partial \varepsilon}u_2 + \frac{\partial N_3}{\partial \varepsilon}u_3 + \frac{\partial N_4}{\partial \varepsilon}u_4$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial N_1}{\partial \eta}u_1 + \frac{\partial N_2}{\partial \eta}u_2 + \frac{\partial N_3}{\partial \eta}u_3 + \frac{\partial N_4}{\partial \eta}u_4$$

$$\frac{\partial v}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon}v_1 + \frac{\partial N_2}{\partial \varepsilon}v_2 + \frac{\partial N_3}{\partial \varepsilon}v_3 + \frac{\partial N_4}{\partial \varepsilon}v_4$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial N_1}{\partial \eta}v_1 + \frac{\partial N_2}{\partial \eta}v_2 + \frac{\partial N_3}{\partial \eta}v_3 + \frac{\partial N_4}{\partial \eta}v_4$$

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Substitute the values of equations (i) to (viii) in equations we get

$$\frac{\partial u}{\partial \varepsilon} = \frac{1}{4}\left[-(1-\eta)\,u_1 + (1-\eta)\,u_2 + (1+\eta)\,u_3 - (1+\eta)\,u_4\right]$$

$$\frac{\partial u}{\partial \eta} = \frac{1}{4}\left[-(1-\varepsilon)\,u_1 - (1+\varepsilon)\,u_2 + (1+\varepsilon)\,u_3 + (1-\varepsilon)\,u_4\right]$$

$$\frac{\partial v}{\partial \varepsilon} = \frac{1}{4}\left[-(1-\eta)\,v_1 + (1-\eta)\,v_2 + (1+\eta)\,v_3 - (1+\eta)\,v_4\right]$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}\left[-(1-\varepsilon)\,v_1 - (1+\varepsilon)\,v_2 + (1+\varepsilon)\,v_3 + (1-\varepsilon)\,v_4\right] \quad \ldots$$

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Writing in the matrix form, we get

$$\begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \varepsilon} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \ldots$$

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Substituting the equation in equation

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times$$

$$\frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

$$\times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \ldots (\quad)$$

ISOPARAMETRIC FORMULATION

i.e., $\{\epsilon\} = [B]\{u\}$

Where $[B]$ – strain - displacement matrix

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times$$

$$\frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

... (   ))

and $|J| = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix}$

ISOPARAMETRIC FORMULATION

Stress - Strain relationship matrix

The element stresses for two dimensional element is given by

$$\{\sigma\} = [D]\{\epsilon\}$$
$$= [D][B]\{u\}$$

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Where $\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$ – Element stresses

$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$ – Element strains

$[D]$ = stress - strain relationship matrix

i.e., For plane stress condition

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

For plane - strain condition

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

Where, E – Young's Modulus

μ – Poisson's ratio

and $[B]$ – Strain - displacement matrix

$\{N\}$ – Nodal displacement vector

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Stiffness matrix for isoparametric quadrilateral element

We know that element stiffness matrix for any element

$$[K] = \int_v [B]^T [D] [B] \, dv$$

For isoparametric quadrilateral element

$$[K] = t \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] |J| \, d\varepsilon \, d\eta$$

Where, t – Thickness of the element

$|J|$ – Determinant of Jacobian matrix

The size of the element stiffness matrix is (8 × 8)

Element force vector

The element force vector is given by

$$\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

Where, N is the shape function

F_x is a load (or) force on x direction

F_y is a force on y direction

Thank You