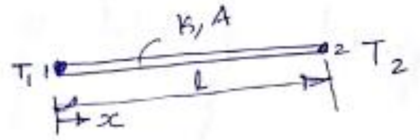
Derivation for conduction stiffness matrix

$$[K] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



T_1, T_2 — Nodal Temperature at the respective nodes.

K - Thermal conductivity of material

Stiffness matrix $[K] = \int_v [B]^T[D][B]\,dv$

Temperature function, $T = N_1T_1 + N_2T_2$

$$N_1 = \frac{l-x}{l} \qquad N_2 = \frac{x}{l}$$

Strain - displacement matrix

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \quad B = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \quad B^T = \begin{bmatrix} -1/l \\ 1/l \end{bmatrix}$$

In one dimensional heat conduction problems

$[D] = [K] = K =$ Thermal Conducivity

Substitute $[B], [B]^T$ and $[D]$ values in Stiffness matrix equation.

Stiffness matrix for heat conduction.

$$[K_c] = \int_0^l \begin{Bmatrix} -\frac{1}{l} \\ 1/l \end{Bmatrix} K \begin{bmatrix} -1/l & \frac{1}{l} \end{bmatrix} dv$$

HEAT TRANSFER - *Stiffness matrix for 1-D heat conduction with free end convection, with internal heat generation*

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & -\frac{1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix} K\,A\,dx \qquad dv = A\,dx$$

$$= KA \begin{bmatrix} \frac{1}{l^2} & -\frac{1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l \qquad \left[\because \int_0^l dx = [x]_0^l\right]$$

$$= KA \begin{bmatrix} \frac{1}{l^2} & -\frac{1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix} [l-0]$$

$$= \frac{KA\,l}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

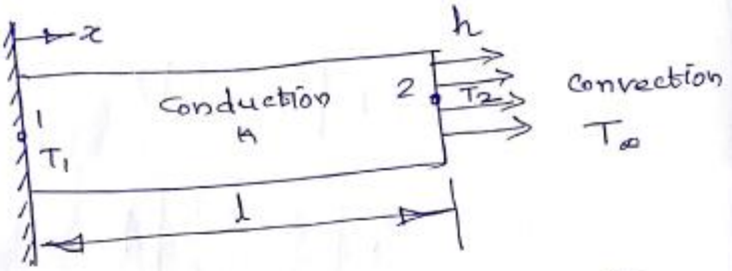
$$= \frac{KA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

A - Area of the element, m^2

K - Thermal Conductivity W/mK

l - length of the element.

Derivation of Stiffness matrix with Conduction and free end Convection.



T_1, T_2 – nodal temperature at the respective nodes.

Assume Convection occurs only from the right end of the element

Conduction stiffness matrix $[K_c] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Convection stiffness matrix

$$[K_h]_{end} = \iint h\,[N]^T[N]\,dA$$

h - Heat transfer coefficient, W/m^2K

h - Heat transfer coefficient, W/m^2K

Shape function (factor) $[N] = [N_1 \quad N_2]$

$$= \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

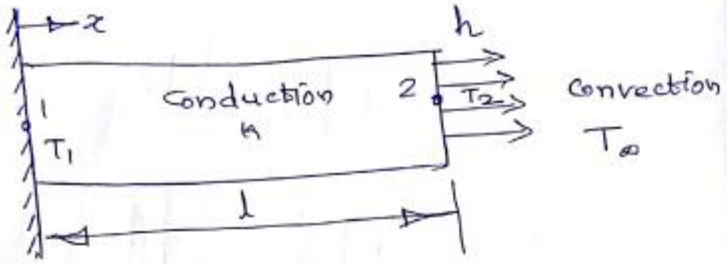
We consider Convection only at the free end

At node 2 $x = l$ $\left[\because N_1 = 0, \; N_2 = 1\right]$

$$[N] = [0 \quad 1]$$

$$[N]^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Derivation of Stiffness matrix with Conduction and free end Convection.



T_1, T_2 – nodal temperature at the respective nodes.

Assume Convection occurs only from the right end of the element

Conduction stiffness matrix $[K] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Convection stiffness matrix

$$[K_h]_{end} = \iint h\, [N]^T [N]\, dA$$

h - heat transfer coefficient, $W/m^2 K$

Shape function (factor) $[N] = [N_1 \quad N_2]$

$$= \left[\frac{l-x}{l} \quad \frac{x}{l}\right]$$

we consider convection only at the free end

At node 2 $x = l$ $\left[\because \begin{matrix} N_1 = 0 \\ N_2 = 1 \end{matrix}\right]$

$$[N] = [0 \quad 1]$$

$$[N]^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

HEAT TRANSFER - *Stiffness matrix for 1-D heat conduction with free end convection, with internal heat generation*

$$[N] \; \& \; [N]^T$$

$$[K_h]_{end} = \iint h \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} [0 \quad 1] \, dA$$

$$= \int_A h \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} dA$$

$$= h \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \int_A dA$$

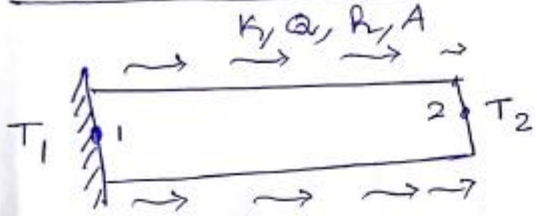
$$= h \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$= hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K] = [K_c] + [K_h]_{end}$$

$$= \frac{kA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Derivation of Stiffness matrix with conduction, convection and internal heat generation.



Consider a rod is subjected to conduction, surface convection and internal heat generation.

K - Thermal Conductivity in W/mK
Q - Internal heat generation in W
h - Convection Coefficient in W/m²K

→ We know that conduction stiffness matrix for one dimensional element is

$$[K_c] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Convection stiffness matrix at element surface is given by

$$[K_h]_s = \iint h [N]^T [N] \, ds$$

$$= hp \int_0^l [N]^T [N] \, dx \qquad [\because ds = p \times dx, \; P - \text{perimeter of the element}]$$

$$= hp \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} dx$$

HEAT TRANSFER - *Stiffness matrix for 1-D heat conduction with free end convection, with internal heat generation*

$$= h_p \int_0^l \begin{bmatrix} \left(\frac{l-x}{l}\right)^2 & \left(\frac{l-x}{l}\right)\cdot\frac{x}{l} \\ \frac{x}{l}\cdot\left(\frac{l-x}{l}\right) & \frac{x}{l}\cdot\frac{x}{l} \end{bmatrix} dx$$

$$= h_p \int_0^l \begin{bmatrix} \left(1-\frac{x}{l}\right)^2 & \frac{x}{l}-\frac{x^2}{l^2} \\ \frac{x}{l}-\frac{x^2}{l^2} & \frac{x^2}{l^2} \end{bmatrix} dx$$

$$= h_p \begin{bmatrix} \left(x+\frac{x^3}{3l^2}-\frac{x^2}{l}\right) & \frac{x^2}{2l}-\frac{x^3}{3l^2} \\ \frac{x^2}{2l}-\frac{x^3}{3l^2} & \frac{x^3}{3l^2} \end{bmatrix}_0^l$$

$$= h_p \begin{bmatrix} l/3 & l/6 \\ l/6 & l/3 \end{bmatrix}$$

$$= \frac{h p l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[K] = [K_c] + [K_h]_s$$

$$[K] = \frac{KA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h p l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Derivation of force matrix for heat transfer problem.

Force matrix due to free end convection.

*(i) Force matrix due to free end convection.

$$\{F_h\}_{end} = h\, T_\infty A \begin{bmatrix} N_1 & \text{at } x = l \\ N_2 & \text{at } x = l \end{bmatrix}$$

$$= h\, T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

(ii) Force matrix due to surface convection and internal heat generation.

$$\{F_h\}_S = \iint h\, T_\infty [N]^T ds = \int h\, T_\infty [N]^T P\, dx$$

$$= \int_0^l h\, T_\infty [N]^T P\, dx$$

$$= P h T_\infty \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= P h T_\infty \begin{Bmatrix} x - \frac{x^2}{2l} \\ \frac{x^2}{2l} \end{Bmatrix}_0^l$$

$$= P h T_\infty \begin{Bmatrix} \left(l - \frac{l^2}{2l}\right) - 0 \\ \left(\frac{l^2}{2l}\right) - 0 \end{Bmatrix}$$

$$= P h T_\infty \begin{Bmatrix} l/2 \\ l/2 \end{Bmatrix}$$

$$\{F_h\}_S = \frac{P h T_\infty l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Force matrix due to internal heat generation is given by

$$\{F_Q\} = \iiint [N]^T Q\, dv = \int_v [N]^T Q\, dv \qquad [\because dv = A \cdot dx$$

$$= \int_0^l [N]^T Q \cdot A \cdot dx$$

$$= Q \cdot A \int_0^l [N]^T dx$$

HEAT TRANSFER - *Stiffness matrix for 1-D heat conduction with free end convection, with internal heat generation*

$$= Q \cdot A \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= Q \cdot A \begin{bmatrix} x - \frac{x^2}{2l} \\ \frac{x^2}{2l} \end{bmatrix}_0^l$$

$$= Q \times A \begin{Bmatrix} \left(l - \frac{l^2}{2l}\right) - 0 \\ \left(\frac{l^2}{2l}\right) - 0 \end{Bmatrix}$$

$$= Q \cdot A \begin{Bmatrix} l/2 \\ l/2 \end{Bmatrix}$$

$$\{F_Q\} = \frac{QAl}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Force matrix, $[F] = \{F_h\}_s + \{F_Q\}$

$$F = \frac{PhT_\infty l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{QAl}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$F = \frac{QAl + PhT_\infty l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

HEAT TRANSFER - *Stiffness matrix for 1-D heat conduction with free end convection, with internal heat generation*

Finite Element equation for heat transfer problems.

(i) writing finite element equation for one dimensional heat conduction problem

$$\{F\} = [K_c]\{T\}$$

$$\frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

(ii) writing finite element equation for conduction with free end convection

$$\{F\} = [K][T]$$

$$\left\{\frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = hT_\infty A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

(iii) writing finite element equation for one dimension element with conduction, surface convection and internal heat generation.

$$\left[\frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{QAl + PhT_\infty l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

A - Area of the element, P - perimeter
K - Thermal conductivity of the element, Q - Heat generating
l - length of the element, T - temperature
h - convection coefficient, T_∞ - Fluid temp