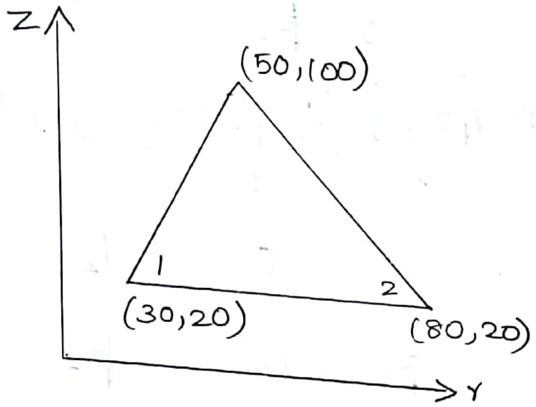
AXISYMMETRIC TRIANGULAR ELEMENT PROBLEM.

For an axisymmetric triangular element as shown in figure. Evaluate the stiffness matrix. Take modulus of elasticity $E = 210$ GPa, Poisson's ratio $\mu = 0.25$. The co-ordinates are given in millimeters.



For the axisymmetric triangular element, the stiffness matrix is

$$[k] = 2\pi r A [B]^T [D] [B]$$

where $r = \dfrac{r_1 + r_2 + r_3}{3}$

$[B]$ = Strain displacement matrix

$[D]$ = Stress - Strain relationship matrix

A = Area of the triangle.

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$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 30 & 20 \\ 1 & 80 & 20 \\ 1 & 50 & 100 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(8000-1000)-30(100-20)+20(50-80)\right]$$

$$= 2000 \text{ mm}^2$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \left(\frac{\alpha_1+\beta_1 r+\gamma_1 z}{r}\right) & 0 & \left(\frac{\alpha_2+\beta_2 r+\gamma_2 z}{r}\right) & 0 & \left(\frac{\alpha_3+\beta_3 r+\gamma_3 z}{r}\right) & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Now

$\alpha_1 = r_2 z_3 - r_3 z_2 = (80\times100)-(50\times20) = 7000$

$\alpha_2 = r_3 z_1 - r_1 z_3 = (50\times20)-(30\times100) = -2000.$

$\alpha_3 = r_1 z_2 - r_2 z_1 = (30\times20)-(80\times20) = -1000.$

$\beta_1 = z_2 - z_3 = 20-100 = -80.$

$\beta_2 = z_3 - z_1 = (100-20) = 80.$

$\beta_3 = z_1 - z_2 = 20-20 = 0.$

$\gamma_1 = r_3 - r_2 = (50-80) = -30$

$\gamma_2 = r_1 - r_3 = (30-50) = -20.$

$\gamma_3 = r_2 - r_1 = (80-30) = 50.$

The coordinates

$$r = \frac{r_1+r_2+r_3}{3} = \frac{30+80+50}{3} = \frac{160}{3}$$

$$z = \frac{z_1+z_2+z_3}{3} = \frac{20+20+100}{3} = \frac{140}{3}$$

and the terms

$$\frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} = \frac{3}{160}\left[7000 - \left(80\times\frac{160}{3}\right) - \left(30\times\frac{140}{3}\right)\right] = 25$$

$$\frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} = \frac{3}{160}\left[-2000 + \left(80\times\frac{160}{3}\right) - \left(20\times\frac{140}{3}\right)\right] = 25$$

$$\frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} = \frac{3}{160}\left[-1000 + 0 + \left(50\times\frac{140}{3}\right)\right] = 25$$

Substituting the above values in strain-displacement matrix, we get

$$[B] = \frac{1}{4000}\begin{bmatrix} -80 & 0 & 80 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & -30 & 0 & -20 & 0 & 50 \\ -30 & -80 & -20 & 80 & 50 & 0 \end{bmatrix}$$

$$= \frac{1}{800}\begin{bmatrix} -16 & 0 & 16 & 0 & 0 & 0 \\ 5 & 0 & 5 & 0 & 5 & 0 \\ 0 & -6 & 0 & -4 & 0 & 10 \\ -6 & -16 & -4 & 16 & 10 & 0 \end{bmatrix}$$

and the transpose of [B] is given by

$$[B]^T = \frac{1}{800}\begin{bmatrix} -16 & 5 & 0 & -6 \\ 0 & 0 & -6 & -16 \\ 16 & 5 & 0 & -4 \\ 0 & 0 & -4 & 16 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

The stress strain relationship matrix is given by

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ \mu & (1-\mu) & \mu & 0 \\ \mu & \mu & (1-\mu) & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix}$$

$$= \frac{2 \cdot 1 \times 10^5}{(1+0\cdot25)(1-0\cdot5)}\begin{bmatrix} (1-0\cdot25) & 0\cdot25 & 0\cdot25 & 0 \\ 0\cdot25 & (1-0\cdot25) & 0\cdot25 & 0 \\ 0\cdot25 & 0\cdot25 & (1-0\cdot25) & 0 \\ 0 & 0 & 0 & \left(\frac{1-0\cdot5}{2}\right) \end{bmatrix}$$

$[D] = 336 \times$

$$[D] = 336 \times 10^3 \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= 336 \times 10^3 \times 0.25 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K] = 2\pi r A [B]^T [D] [B]$$

$$= 2\pi \times \frac{160}{3} \times 2000 \times \frac{1}{800} \begin{bmatrix} -16 & 5 & 0 & -6 \\ 0 & 0 & -6 & -16 \\ 16 & 5 & 0 & -4 \\ 0 & 0 & -4 & 16 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix} \times 84 \times 10^3$$

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \frac{1}{800} \begin{bmatrix} -16 & 0 & 16 & 0 & 0 & 0 \\ 5 & 0 & 5 & 0 & 5 & 0 \\ 0 & -6 & 0 & -4 & 0 & 10 \\ -6 & -16 & -4 & 16 & 10 & 0 \end{bmatrix}$$

$$= \frac{2\pi \times 160 \times 2000 \times 84 \times 10^3}{3 \times 800 \times 800} \begin{bmatrix} -16 & 5 & 0 & -6 \\ 0 & 0 & -6 & -16 \\ 16 & 5 & 0 & -4 \\ 0 & 0 & -4 & 16 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} -43 & -6 & 53 & -4 & 5 & 10 \\ -1 & -6 & 31 & -4 & 15 & 10 \\ -11 & -18 & 21 & -12 & 5 & 30 \\ -6 & -16 & -4 & 16 & 10 & 0 \end{bmatrix}$$

Result :-

The stiffness matrix for the axisymmetric triangular element is.

$$[K] = 87965 \begin{bmatrix} 719 & 162 & -669 & -52 & -65 & -110 \\ 162 & 364 & -62 & -184 & -190 & -180 \\ -669 & -62 & 1019 & -148 & 115 & 210 \\ -52 & -184 & -148 & 304 & 140 & -120 \\ -65 & -190 & 115 & 140 & 175 & 50 \\ -110 & -180 & 210 & -120 & 50 & 300 \end{bmatrix} \text{ N/mm}$$