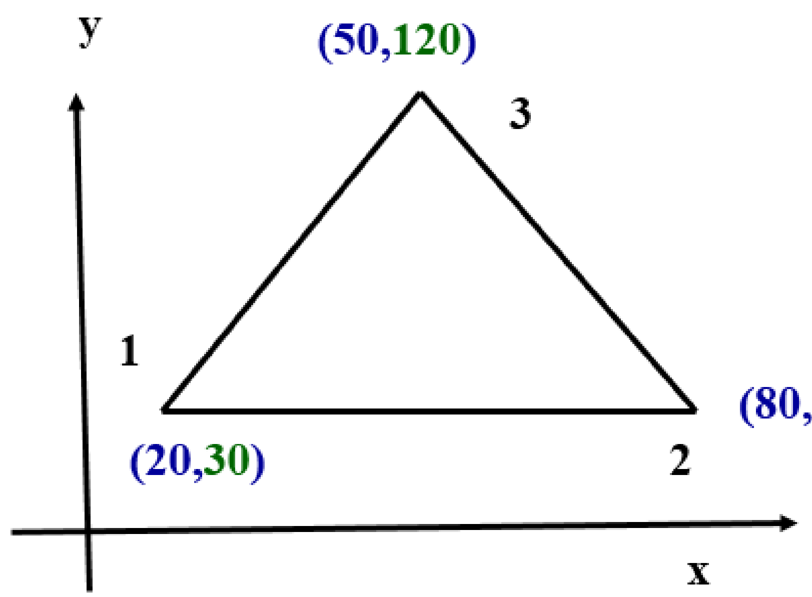
Finite Element Analysis
Two-Dimensional Scalar Variable Problems

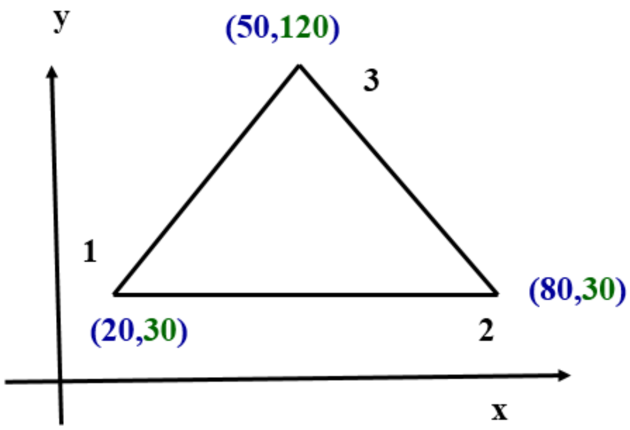
Evaluate the stiffness matrix for the elements shown in Figure. The coordinates are given in units of millimetres. Assume plane stress conditions. Let $E = 210GPa$, $\mu = 0.25mm$, and t =10 mm. For the elements given in Figure, the nodal displacements are given as $u_1 = 2.0mm$, $v_1 = 1.0mm$, $u_2 = 0.5mm$, $v_2 = 0.0mm$, $u_3 = 3.0mm$, $v_3 = 1.0mm$ .Determine the element stresses $\sigma_x, \sigma_y, \tau_{xy}, \sigma_1$, and σ_2 and the principal angle θ_p.



Constant Strain Triangular element [CST] Stiffness matrix [B]

- $(x_1, y_1) = (20,30)$
  $(x_2, y_2) = (80,30)$
  $(x_3, y_3) = (50,120)$

$To\ find;\ [K] = [B]^T[D][B].A.t$



Step i) Area of Triangular Element

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = A = \frac{1}{2}\begin{bmatrix} 1 & 20 & 30 \\ 1 & 80 & 30 \\ 1 & 50 & 120 \end{bmatrix}$$

$$= \frac{1}{2}\Big(1(80 \times 120 - 30 \times 50) - 20(1 \times 120 - 30 \times 1) + 30(1 \times 50 - 80 \times 1)\Big)$$

$$A = 2700mm^2$$

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$$= \frac{1}{2}\left(1(80\times120-30\times50)-20(1\times120-30\times1)+30(1\times50-80\times1)\right)$$

$$= 2700mm^2$$

Constant Strain Triangular element [CST] Stiffness matrix [B]

Step ii) Strain Displacement matrix

- $$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$$q_1 = y_2 - y_3 = 30 - 120 = -90 \qquad r_1 = x_3 - x_2 = 50 - 80 = -30$$

$$q_2 = y_3 - y_1 = 120 - 30 = 90 \qquad r_2 = x_1 - x_3 = 20 - 50 = -30$$

$$q_3 = y_1 - y_2 = 30 - 30 = 0 \qquad r_3 = x_2 - x_1 = 80 - 20 = 60$$

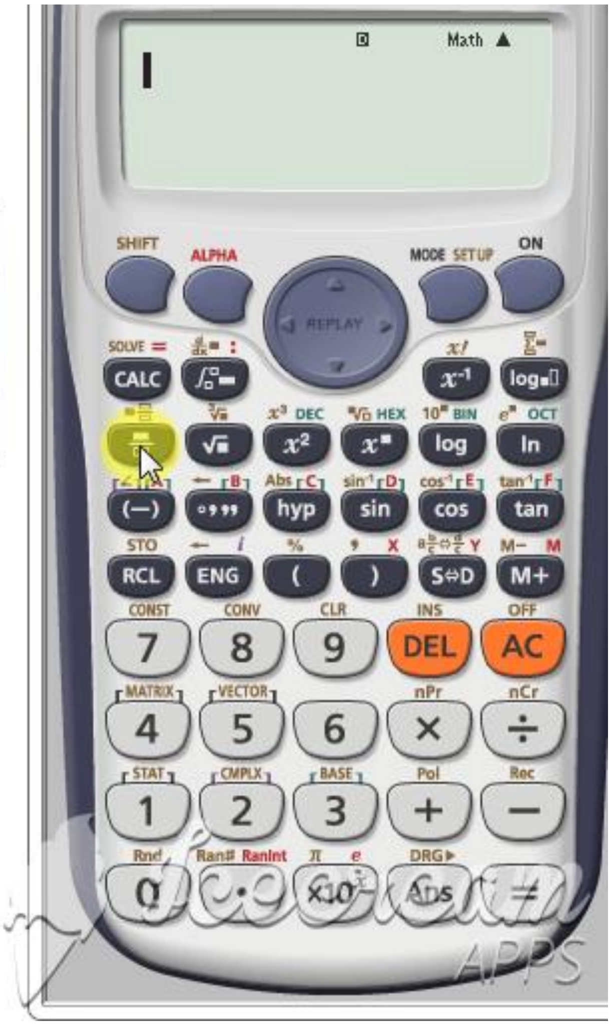
$$[B] = \frac{1}{2 \times 2700}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{2 \times 2700} \times [30]\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{180}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Constant Strain Triangular element [CST] Stiffness matrix [B]

$$[B]=\frac{1}{2\times 2700}\times[30]\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\ 0 & -1 & 0 & -1 & 0 & 2\\ -1 & -3 & -1 & 3 & 2 & 0\end{bmatrix}$$



Constant Strain Triangular element [CST] Stiffness matrix [B]

- $$[B]^T = \frac{1}{180}\begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & 3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

ii) Stress − strain matrix [D] for plane stress condition: $[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$

$$[D] = \frac{2.1 \times 10^5}{1 - 0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix} \Rightarrow \mathrm{D} = \frac{2.1 \times 10^5}{1 - 0.25^2}[0.25]\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[\mathrm{D}] = 56000\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Constant Strain Triangular element [CST] Stiffness matrix [B]

- $[K] = [B]^T[D][B].A.t$

$$= [D][B]$$

$$= 56 \times 10^3 \times \frac{1}{180}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Matrix A[3x3] *Matrix B[3x3]* *Matrix C[3x3]*

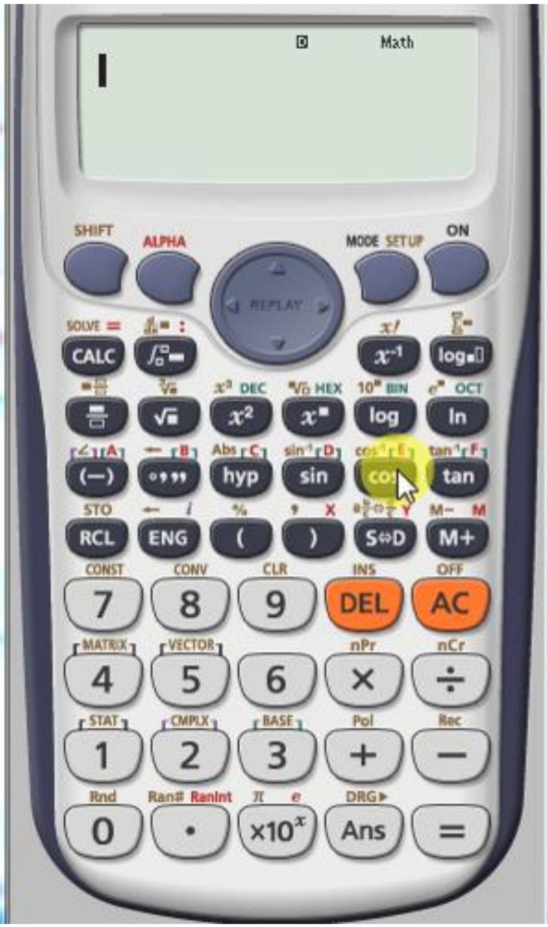
$$= 311.111\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Matrix A[3x3]x Matrix B[3x3]

Matrix A[3x3]x Matrix C[3x3]

$$= 311.111\begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

Constant Strain Triangular element [CST] Stiffness matrix [B]



- $[K] = [B]^T[D][B].A.t$

$$= [D][B]$$

$$= 56 \times 10^3 \times \frac{1}{180}\begin{bmatrix}4 & 1 & 0\\1 & 4 & 0\\0 & 0 & 1.5\end{bmatrix}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\0 & -1 & 0 & -1 & 0 & 2\\-1 & -3 & -1 & 3 & 2 & 0\end{bmatrix}$$

Matrix A[3x3] *Matrix B[3x3]* *Matrix C[3x3]*

$$= 311.111\begin{bmatrix}4 & 1 & 0\\1 & 4 & 0\\0 & 0 & 1.5\end{bmatrix}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\0 & -1 & 0 & -1 & 0 & 2\\-1 & -3 & -1 & 3 & 2 & 0\end{bmatrix}$$

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$$To\ find; [K] = [B]^T [D][B].A.t$$

- $= [B]^T [D][B].$

$$= \frac{1}{180} \times 311.111 \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

$$= 1.72839 \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

Constant Strain Triangular element [CST] Stiffness matrix [B]

$$= 1.72839 \begin{array}{c} mat\ A^1 \\ \\ \\ \\ \\ mat\ A^2 \end{array} \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{array}{c} mat\ B \qquad\qquad mat\ C \\ \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} mat\ A^1 \times mat\ B & mat\ A^1 \times mat\ C \\ mat\ A^2 \times mat\ B & mat\ A^2 \times mat\ C \end{bmatrix}$$

$$= A.t[B]^T[D][B]$$

$$= 46666.53 \begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix}$$

Constant Strain Triangular element [CST] Stiffness matrix [B]

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$$= A.t[B]^T[D][B]$$

$$= 1.72839 \begin{array}{c} mat\ A^1 \\ \left[\begin{matrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{matrix}\right] \\ mat\ A^2 \end{array} \begin{array}{cc} mat\ B & mat\ C \\ \multicolumn{2}{c}{\left[\begin{matrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{matrix}\right]} \end{array}$$

$$\begin{bmatrix} mat\ A^1 \times mat\ B & mat\ A^1 \times mat\ C \\ mat\ A^2 \times mat\ B & mat\ A^2 \times mat\ C \end{bmatrix}$$

$$= 46666.53 \begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix}$$

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Constant Strain Triangular element [CST] Stiffness matrix [B]

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$$= 1.72839 \begin{array}{c} \text{mat } A^1 \\ \\ \\ \\ \\ \text{mat } A^2 \end{array} \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{array}{cc} \text{mat } B & \text{mat } C \end{array} \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \text{mat } A^1 \times \text{mat } B & \text{mat } A^1 \times \text{mat } C \\ \text{mat } A^2 \times \text{mat } B & \text{mat } A^2 \times \text{mat } C \end{bmatrix}$$

$$= A.t[B]^T[D][B]$$

$$= 46666.53 \begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix}$$

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THANK YOU