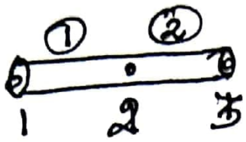
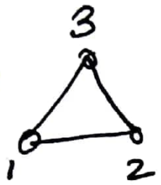
unit-3 Two dimensional problem

1D [line]

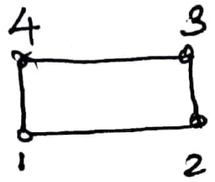


[Spring, Bar, Truss, Beam, pipe, etc]

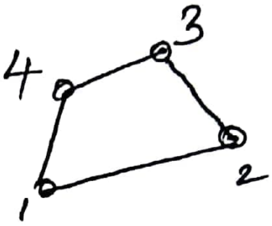
2D [plane]



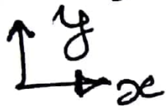
Triangle



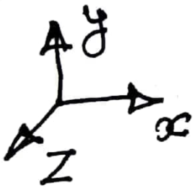
Rectangle

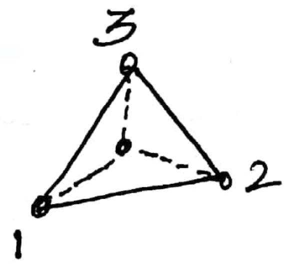


Quadrilateral

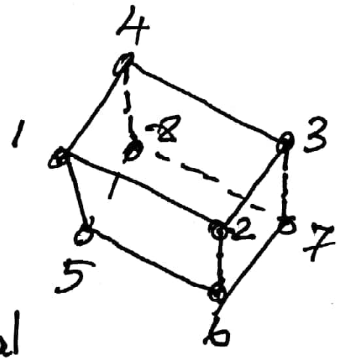


3D [Solid]

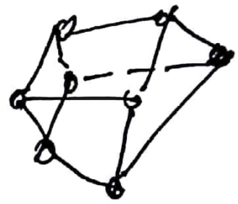




Tetrahedral



Regular hexahedral



Irregular hexahedral

The displacement Vector "u" is given by $\begin{Bmatrix} u \\ v \end{Bmatrix}$

$$U = \begin{Bmatrix} u \\ v \end{Bmatrix} = [u \quad v]^T$$

Where u, v are x & y components of u

2D problem

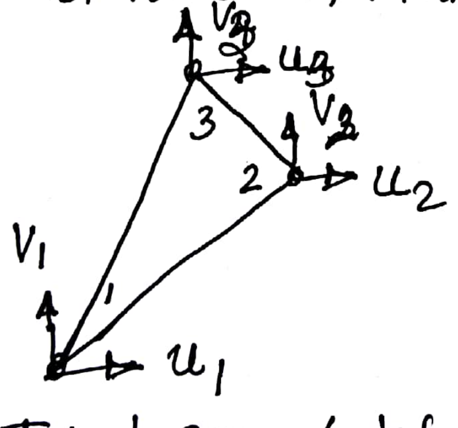
Stress & Strain are given by

$$\{\sigma\} = \begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{Bmatrix}. \qquad \epsilon = \begin{Bmatrix}\epsilon_x \\ \epsilon_y \\ \gamma_{xy}\end{Bmatrix}$$

$$\{\sigma\} = [\underset{E}{D}]\{\epsilon\} \qquad \boxed{\sigma = E\epsilon}$$

$$E = \frac{\sigma}{\epsilon},$$ Hook's law

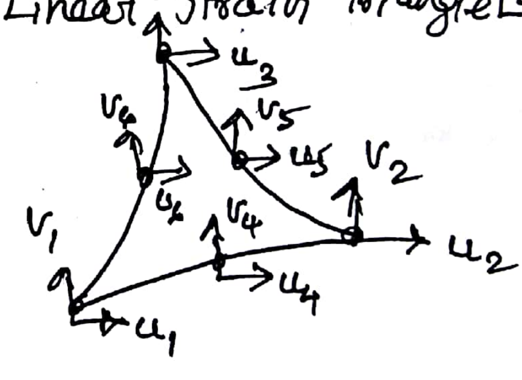
Constant Strain Triangle [CST]



Total 3×2 = 6 dof

Edges are straight.

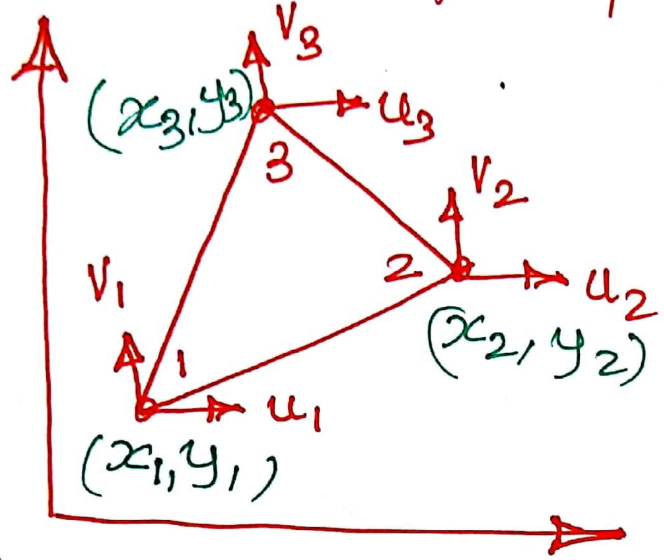
Linear Strain Triangle [LST]



Totally 6×2 = 12 dof.

Edges are Curved.

Derivation of shape function for CST



Let the nodal displacements be

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

u_1, u_2, u_3 → displacement along x axis.

v_1, v_2, v_3 → displacement along y axis

Since a CST has 3 nodes and each node has 2 dof, totally 6 dof. Hence it should be approximated with 6 gennalized Co-ordinate. (a_1 to a_6).

$$u = a_1 + a_2 x + a_3 y$$

$$V = a_4 + a_5 x + a_6 y$$

Now, apply the nodal conditions.

$u_1 = a_1 + a_2 x_1 + a_3 y_1$ $\quad v_1 = a_4 + a_5 x_1 + a_6 y_1$

$u_2 = a_1 + a_2 x_2 + a_3 y_2$ $\quad v_2 = a_4 + a_5 x_2 + a_6 y_2$

$u_3 = a_1 + a_2 x_3 + a_3 y_3$ $\quad v_3 = a_4 + a_5 x_3 + a_6 y_3$

In matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$x = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$x^{-1} = \frac{C^T}{|x|}$$

$$x^{-1} = \frac{\begin{bmatrix} (x_2 y_3 - x_3 y_2) & -(y_3 - y_2) & (x_3 - x_2) \\ -(x_1 y_3 - x_3 y_1) & (y_3 - y_1) & -(x_3 - x_1) \\ (x_1 y_2 - x_2 y_1) & -(y_2 - y_1) & (x_2 - x_1) \end{bmatrix}^T}{(x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)} \Rightarrow 2A$$

2D- problem

$$x^{-1} = \frac{1}{2A}\begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$U = [1 \; x \; y]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [1 \; x \; y]\frac{1}{2A}\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{p_1 + q_1x + r_1y}{2A} & \frac{p_2 + q_2x + r_2y}{2A} & \frac{p_3 + q_3x + r_3y}{2A} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

This is form of.

$$u = [N_1 \; N_2 \; N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{|||ly} \qquad V = [N_1 \; N_2 \; N_3]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where

$$N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}$$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

$p_1 = x_2 y_3 - x_3 y_2$ $\quad q_1 = y_2 - y_3$ $\quad r_1 = x_3 - x_2$

$p_2 = x_3 y_1 - x_1 y_3$ $\quad q_2 = y_3 - y_1$ $\quad r_2 = x_1 - x_3$

$p_3 = x_1 y_2 - x_2 y_1$ $\quad q_3 = y_1 y_2$ $\quad r_3 = x_2 - x_1$