1. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$. ~~Show that~~ Find the displacement.

Soln. :

The ODWE is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are,

i). $y(0,t)=0, \quad \forall t$

ii). $y(l,t)=0, \quad \forall t$

iii). $y(x,0)=0, \quad \forall x$

iv). $\frac{\partial}{\partial t} y(x,0) = \lambda x(l-x)$

The suitable solution is,

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow (1)$$

Applying condn. (i) in (1), we get

$y(0,t)=0$

$(A(1)+B(0))(C\cos pat + D\sin pat) = 0$

$A(C\cos pat + D\sin pat) = 0$

Here $C\cos pat + D\sin pat \neq 0$ ($\because$ It is a fn. of 't')

$$\boxed{A=0}$$

$(1) \Rightarrow y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow (2)$

Applying condn. (ii) in (2), we get

$$y(l, t) = 0$$

$$B \sin pl \, (C \cos pat + D \sin pat) = 0$$

Here $C \cos pat + D \sin pat \neq 0$ ($\because$ It is a fn. of 't')

and $B \neq 0$ (If $B = 0$, then we get a trivial soln.)

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

$$(2) \Rightarrow y(x, t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right) \rightarrow (3)$$

Applying condn. (iii) in (3),

$$y(x, 0) = 0$$

$$B \sin \frac{n\pi x}{l} \left(C(1) + D(0) \right) = 0$$

$$BC \sin \frac{n\pi x}{l} = 0$$

Here $B \neq 0$ (Already explained)

$$\sin \frac{n\pi x}{l} \neq 0$$

$$\boxed{C = 0}$$

$$(3) \Rightarrow y(x, t) = B \sin \frac{n\pi x}{l} \left[0 + D \sin \frac{n\pi at}{l} \right]$$

$$= BD \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$$

$$= B \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \quad \text{where } B = BD$$

The most general soln. is,

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \rightarrow (4)$$

Before applying condn. (iv), differentiate (4) partially w.r.t 't', we get

$$\frac{\partial}{\partial t} y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)$$

Applying condn. iv)

$$\frac{\partial}{\partial t} y(x,0) = \lambda x(l-x)$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right) = \lambda(lx-x^2)$$

HRSS $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \lambda(lx-x^2)$ where $b_n = B_n\left(\frac{n\pi a}{l}\right)$

To find b_n:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \lambda(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\frac{-l}{n\pi}(lx-x^2)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l-2x)\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left(0-0-\frac{2l^3}{n^3\pi^3}(-1)^n\right)-\left(0-0-\frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2\lambda}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n+\frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2\lambda}{l}\frac{2l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n\frac{n\pi a}{l} = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{l}{n\pi a}\frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4\lambda l^3}{n^4\pi^4 a}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} \frac{8\lambda l^3}{n^4\pi^4 a}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$(5) \Rightarrow y(x,t) = \sum_{n=odd}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \sum_{n=odd}^{\infty}\frac{8\lambda l^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{\pi^4 a}\sum_{n=odd}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$