2] A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, then find the displacement.

Soln.

The ODWE is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

i). $y(0,t) = 0, \quad \forall t$

ii). $y(l,t) = 0, \quad \forall t$

iii). $\frac{\partial}{\partial t} y(x,0) = 0, \quad \forall x$

iv). $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$

The suitable solution is,

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \rightarrow (1)$$

Applying i) in (1), we get

$y(0,t) = 0$

$(A(1) + B(0))(C \cos pat + D \sin pat) = 0$

$A \, (C \cos pat + D \sin pat) = 0$

$C \cos pat + D \sin pat \neq 0$ (It is a fn of 't')

$\Rightarrow \boxed{A=0}$

Subs. $A=0$ in (1),

$$y(x,t) = B \sin px \,(C \cos pat + D \sin pat) \rightarrow (2)$$

Applying ii) in (2),

$$y(l,t)=0$$

$$B \sin pl \,(C \cos pat + D \sin pat) = 0$$

$B \neq 0$ (If $B=0$, we get a trivial soln.)

$C \cos pat + D \sin pat \neq 0$ ($\because$ it is a fn. of 't')

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Subs. $p = \frac{n\pi}{l}$ in (2),

$$y(x,t) = B \sin \frac{n\pi x}{l}\left[C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l}\right] \rightarrow (3)$$

Before applying iii)., differentiate partially (3) w.r.t 't'

$$\frac{\partial}{\partial t} y(x,t) = B \sin \frac{n\pi x}{l}\left[-C \sin \frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) + D \cos \frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Applying condition iii)., we get

$$\frac{\partial}{\partial t} y(x,0) = 0$$

$$B \sin \frac{n\pi x}{l}\left[0 + \frac{n\pi a}{l} D\right] = 0$$

$$B D \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = 0$$

Here $B \neq 0$ [If $B=0$, we get a trivial soln.]

$\sin \frac{n\pi x}{l} \neq 0$ [$\because$ It is a fn. of x]

$\Rightarrow \boxed{D=0}$

Subs $D=0$ in (3),

$$y(x,t) = B \sin\frac{n\pi x}{l}\left(C \cos\frac{n\pi at}{l}\right)$$

$$= BC \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

The most general soln. is,

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \rightarrow (4)$$

Applying condition iv) in (4), we get

$$y(x,0) = y_0 \sin^3\frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$\Rightarrow B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + \cdots$$

$$= \frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}$$

Equating the like coefficients, we get

$$B_1 = \frac{3y_0}{4}; \quad B_2 = 0; \quad B_3 = -\frac{y_0}{4}; \quad B_4 = B_5 = \cdots = 0$$

Subs the above values in (4),

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

Displacing position

Hw I. $y(x,0) = k \sin\frac{3\pi x}{l} \cos\frac{2\pi x}{l}$

2] a sinusoidal arc of height y_0

$\hookrightarrow y_0 \sin\frac{\pi x}{l}$