Index p = no of +ve square terms in canonical form

= 2

Signature = Diff b/w +ve square terms and -ve square terms.

= 2 - 0 = 2

Example: 4

Reduce the quadratic to canonical form of $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ and discuss about its nature.

Sol

The matrix form of the quadratic eqn is

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1:

Find C.E, EV & EVc

The characteristic equation is

$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

D_1 = Sum of the leading diagonal elements

$= 1 + 2 + 1$

D_2 = Sum of the minor of the leading "

$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$

$= (2-1) + (1) + (2-1)$

$= 3$

$D_3 = |A|$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) + 1(-1+0)$$

$$= 1 - 1 = 0$$

C.E is $\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda = 0, \quad \lambda = 1, \quad \lambda = 3$$

3

$-1 \quad -3$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

Eigen vector

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case i) $\lambda = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{-1+0} = \frac{x_2}{0-1} = \frac{x_3}{2-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

when $\lambda = 3$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -2 & -1 & 0 & -2 \\ -1 & -1 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{0+2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

when $\lambda = 1$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{-0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$X_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Step 2 Check orthogonal condition

$$x_1^T x_2 = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$x_2^T x_3 = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$x_3^T x_1 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

Step 3 : Find normalized vector

Eigenvector $\quad l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad$ Normalized vector

$$x_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad l(x_1) = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \qquad N_1 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \qquad l(x_2) = \sqrt{(-1)^2 + 0 + (-1)^2} = \sqrt{2}$$

$$x_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Step 4 Model Matrix

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

Step 5: $N^T A N = D$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Step 6:

Canonical form

$Y^T D Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore$ The canonical form is $0y_1^2 + y_2^2 + 3y_3^2$

$\therefore$ The nature of the quadratic form is positive semi definite.

Rank $(r) = 2$

Index $(p) = 2$

signature $(s) = 2$

Example: 5:

Reduce the quadratic to canonical form $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ and discuss about its nature.